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ON THE CHARACTERIZATION OF FINITE DIMENSIONAL HIDA DISTRIBUTIONS

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Abstract

The mathematical framework of white noise analysis is based on an infinite dimensional analogue of the Schwartz distribution theory. The Lebesgue measure on $\mathbb{R}^k$ is replaced with standard Gaussian measures μ on infinite dimensional spaces. There is an infinite dimensional analogue $(\mathcal{E}) \subset L^2(\mathcal{E}^*, \mu) \subset (\mathcal{E})^*$ of a Gel'fand triple $\mathcal{E} \subset E \subset \mathcal{E}^*$ which is obtained from $\mathcal{S}(\mathbb{R}^k) \subset L^2(\mathbb{R}^k) \subset \mathcal{S}^*(\mathbb{R}^k)$ in a general setup. There are spaces $(\mathcal{E}^\beta), (\mathcal{E}^\beta)^*$, $\beta \in [0, 1)$ with $(\mathcal{E}^\beta) \subset (\mathcal{E}) \subset L^2(\mathcal{E}^*, \mu) \subset (\mathcal{E})^* \subset (\mathcal{E}^\beta)^*$.

The compositions of Schwartz distributions and Gaussian random variables have been discussed. A new Gel'fand triple $\mathcal{H}(\mathbb{R}^k) \subset \mathcal{H}_0(\mathbb{R}^k) \subset \mathcal{H}^*(\mathbb{R}^k)$ over $\mathbb{R}^k$ plays the key role for characterizing the class of functions F such that the composition $F \circ (\langle \cdot, \xi_1 \rangle, \ldots, \langle \cdot, \xi_k \rangle)$, $\xi_1, \ldots, \xi_k \in E$ is in $(\mathcal{E})^*$ and the class of functions F such that $F \circ (\langle \cdot, \xi_1 \rangle, \ldots, \langle \cdot, \xi_k \rangle)$, $\xi_1, \ldots, \xi_k \in E$ is in $(\mathcal{E})$. Moreover, the spaces $\mathcal{H}(\mathbb{R}^k)$, $\mathcal{H}^*(\mathbb{R}^k)$ have been characterized.

This work introduces a new Gel'fand triple $\mathcal{H}^\beta(\mathbb{R}^k) \subset \mathcal{H}_0^0(\mathbb{R}^k) \subset (\mathcal{H}^\beta)^*(\mathbb{R}^k)$ to extend these types of results to $(\mathcal{E}^\beta)$ and $(\mathcal{E}^\beta)^*$ with characterizations of the spaces $\mathcal{H}^\beta(\mathbb{R}^k), (\mathcal{H}^\beta)^*(\mathbb{R}^k)$.

Introduction

Hida [Hi 75] opened up a new area of mathematical research in 1975, called white noise analysis. He has used the white noise $\{\dot{B}(t) = \frac{d}{dt}B(t); t \in \mathbb{R}\}$ as a system of coordinates to represent a Brownian functional $f(B(t); t \in \mathbb{R})$ as a white noise functional $\phi(\dot{B}(t); t \in \mathbb{R})$.

The mathematical framework of white noise analysis is based on an infinite dimensional analogue of the Schwartz distribution theory. The Lebesgue measure on $\mathbb{R}^k$ is replaced with standard Gaussian measures on infinite dimensional spaces.

Kubo and Takenaka [KT 80a] have introduced a Gel'fand triple

$$(\mathcal{E}) \subset (E) \subset (\mathcal{E})^*$$

which induces spaces of test functionals and Hida distributions. A particular choice of this Gel'fand triple

$$(\mathcal{S}) \subset (L^2) \subset (\mathcal{S})^*$$

has been studied by many people ([PS 91], [KPS 91], etc). The space $(\mathcal{S})^*$ of Hida distributions has been characterized in the paper [PS 91] and the space $(\mathcal{S})$ of test functionals of white noise has been characterized in the paper [KPS 91].

Kontratiev and Streit [KS 92] have recently introduced another Gel'fand triple over the white noise space:

$$(\mathcal{S}^\beta) \subset (L^2) \subset (\mathcal{S}^\beta)^*, \qquad \beta \in [0,1).$$

The spaces $(\mathcal{S}^\beta)$ and $(\mathcal{S}^\beta)^*$ have been characterized in the paper [KS 92]. We note that

$$(\mathcal{S}^\beta) \subset (\mathcal{S}) \subset (L^2) \subset (\mathcal{S})^* \subset (\mathcal{S}^\beta)^*, \qquad \beta \in [0,1).$$

One of the important white noise functionals, ϕ is of the form

$$\phi(\dot{B}) = P(\langle \dot{B}, \xi_1 \rangle, \cdots, \langle \dot{B}, \xi_k \rangle)$$

where $\xi_1, \cdots, \xi_k \in \mathcal{S}(\mathbb{R})$, the space of Schwartz test functions and P is a polynomial.

Some interest in studying the relationship between the Gel'fand triple

$$\mathcal{S}(\mathbb{R}^k) \subset L^2(\mathbb{R}^k) \subset \mathcal{S}^*(\mathbb{R}^k)$$

and the infinite dimensional Gel'fand triple

$$(\mathcal{S}) \subset (L^2) \subset (\mathcal{S})^*$$

has risen. This has provided some motivation to study finite dimensional Hida distributions.

The following discusses two types of particular results with respect to the relationship between the Gel'fand triples mentioned above. Compositions of Schwartz distributions and Gaussian random variables have been discussed by some authors ([Kub 83], [Kuo 83], [KK 91], etc). It has been shown in [Kub 83] that the compositions $F \circ (\langle \cdot, \xi_1 \rangle, \cdots, \langle \cdot, \xi_k \rangle)$, $\xi_1, \cdots, \xi_k \in L^2(\mathbb{R})$ are in $(\mathcal{S})^*$ for any $F \in \mathcal{S}^*(\mathbb{R}^k)$. In the article [KK 92], Kubo and Kuo have introduced a new Gel'fand triple

$$\mathcal{H}(\mathbb{R}^k) \subset \mathcal{H}_0(\mathbb{R}^k) \subset \mathcal{H}^*(\mathbb{R}^k)$$

over $\mathbb{R}^k$ and characterized the class of functions F such that the composition $F \circ (\langle \cdot, \xi_1 \rangle, \cdots, \langle \cdot, \xi_k \rangle)$, $\xi_1, \cdots, \xi_k \in L^2(\mathbb{R})$ is in $(\mathcal{S})^*$ with a more general set-up.

Secondly, it has been shown in [KPS 91] that if f is a function on $\mathbb{R}$ such that its Fourier transform $\hat{f}$ is in $L^1(\mathbb{R})$ and has compact support, then the composition $f \circ \langle \cdot, \xi \rangle$ is in $(\mathcal{S})$. However, the condition $f \in \mathcal{S}(\mathbb{R})$ does not imply that $f \circ \langle \cdot, \xi \rangle \in$

$(\mathcal{S})$. For example, $\exp(-\langle\cdot,\xi\rangle^2) \notin (\mathcal{S})$ where $\xi \in \mathcal{S}(\mathbb{R}^k)$. Kubo and Kuo in [KK 92] have characterized such F that $F \circ \langle\cdot,\xi\rangle \in (\mathcal{S})$.

It is natural to extend these two types of results in the paper [KK 92] discussed above to the pair of spaces $(\mathcal{S}^\beta)$ and $(\mathcal{S}^\beta)^*$. This dissertation is based on the paper [KK 92] and extends some of the results to the Gel'fand triple

$$(\mathcal{S}^\beta) \subset (L^2) \subset (\mathcal{S}^\beta)^*, \qquad \beta \in [0,1).$$

We will modify their calculations and the arguments so that they work for the new Gel'fand triple. For this purpose, this dissertation is organized as follows: Chapter one briefly reviews the Hida distributions and test functionals with their characterization theorems. Chapter two also briefly reviews the finite dimensional Hida distributions and state some theorems, including ones which have been mentioned above without proofs. Chapter three introduces new spaces $\mathcal{H}^\beta(\mathbb{R}^k)$, $(\mathcal{H}^\beta)^*(\mathbb{R}^k)$, $\mathcal{F}^\beta(\mathbb{R}^k)$, and $(\mathcal{F}^\beta)^*(\mathbb{R}^k)$ with some properties and characterization theorems of those spaces, and finally, chapter three characterizes finite dimensional Hida distributions of order β.

Chapter 1. Hida Distributions

§1.1. Hida Distributions

For a short description of the space of Hida distributions, the constructions of Gel'fand triples $(\mathcal{E}) \subset (L^2) \subset (\mathcal{E})^*$, which has been introduced by Kubo and Takenaka and $(\mathcal{E}^\beta) \subset (L^2) \subset (\mathcal{E}^\beta)^*$, which is the generalized one of the Gel'fand triple $(\mathcal{S}^\beta) \subset (L^2) \subset (\mathcal{S}^\beta)^*$ will be briefly reviewed in this section.

Let E be a real separable Hilbert space with norm $|\cdot|_0$ and A be a linear operator on E such that there exists an orthonormal basis $\{\zeta_j\}_{j=1}^\infty$ for E satisfying the following three conditions: (1) $A\zeta_j = \lambda_j \zeta_j$, $j = 1, 2, \cdots$; (2) $1 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots$; (3) $\sum_{j=1}^\infty \lambda_j^{-2} < \infty$.
Denote $\rho = \lambda_1^{-1}$ for a later use.

For each $p \in \mathbb{R}$, we define a norm $|\cdot|_p$ on E by

$$|\xi|_p = |A^p \xi|_0.$$

For $p \geq 0$, let E_p be the domain of A^P, i.e.,

$$E_p \equiv \{\xi \in E;\, |A^p\xi|_0 < \infty\}$$

and E_p^* be the dual space of $E_p, p \geq 0$. The space E_{-p} which is defined to be the completion of E with respect to the norm $|\cdot|_{-p}$ can be identified as the dual space of E_p. And it is clear that $E_p \subset E_q$ for any $p \geq q$. Thus, define $\mathcal{E}$ to be the projective limit of the spaces $\{E_p; p \geq 0\}$. Then $\mathcal{E} = \cap_{p \geq 0} E_p$. Let $\mathcal{E}^*$ be the dual space of $\mathcal{E}$. Then $\mathcal{E}^* = \cup_{p \geq 0} E_{-P}$. It can be easily checked by using the conditions (1) and

(3) for the opertor A that for any $p > 0$, the inclusion map from E_{p+r} into E_p is a Hilbert-Schmidt operator if $r > 1$. Therefore, $\mathcal{E}$ is a nuclear space and there is a Gel'fand triple:

$$\mathcal{E} \subset E \subset \mathcal{E}^*.$$

By the Bochner-Minlos theorem there exists a unique probability measure μ on $\mathcal{E}^*$ such that

$$\int_{\mathcal{E}^*} e^{i\langle x,\xi\rangle}\, d\mu(x) = \exp[-\frac{1}{2}|\xi|_0^2], \qquad \xi \in \mathcal{E},$$

where $\langle \cdot,\cdot \rangle$ denotes the pairing between $\mathcal{E}^*$ and $\mathcal{E}$. For an infinite dimensional analogue of the Gel'fand triple $\mathcal{E} \subset E \subset \mathcal{E}^*$, we consider classes of functions on the space $(\mathcal{E}^*, \mu)$. For simplicity, (L^2) denotes the complex Hilbert space $L^2(\mathcal{E}^*, \mu)$ of square integrable functions with respect to the measure μ. By the Wiener-Itô decomposition theorem each element $\phi \in (L^2)$ can be uniquely represented by

$$\phi = \sum_{n=0}^{\infty} I_n(f_n), \qquad f_n \in E_{\mathbb{C}}^{\widehat{\otimes} n}$$

where I_n is the multiple Wiener integral of order n and $E_{\mathbb{C}}^{\widehat{\otimes} n}$ denotes the n-th symmetric tensor product of the complexification of E. Moreover, for $\phi \in (L^2)$

$$\|\phi\|_0^2 = \sum_{n=0}^{\infty} n! |f_n|_0^2$$

where $\|\cdot\|_0$ is the norm of the space (L^2) and $|\cdot|_0$ is the norm of each space $E_{\mathbb{C}}^{\widehat{\otimes} n}$. Now, in order to construct Gel'fand triples which were mentioned in the beginning of this section, the second quantization $\Gamma(A)$ of the operator A which is densely defined on (L^2) by: for $\phi = \sum_{n=0}^{\infty} I_n(f_n) \in (L^2)$

$$(\Gamma(A)\phi)(x) \equiv \sum_{n=0}^{\infty} I_n(A^{\otimes n} f_n)$$

is used. For each integer p, define a norm $\|\cdot\|_p$ on (L^2) by

$$\|\phi\|_p \equiv \|\Gamma(A)^p\phi\|_0.$$

Then for $\phi = \sum_{n=0}^{\infty} I_n(f_n)$ in (L^2)

$$\|\phi\|_p^2 = \sum_{n=0}^{\infty} n!|(A^p)^{\otimes n} f_n|_0^2\,.$$

From the fact that $\|\phi\|_0 \leq \|\phi\|_p$ for $p \geq 0$, let $(E_p), p \geq 0$ denote the domain of $\Gamma(A)^p$ which is given by

$$\{\phi = \sum_{n=0}^{\infty} I_n(f_n) \in (L^2)\,;\ \|\ \phi\ \|_p < \infty\}.$$

From the fact that $\|\phi\|_{-p} \leq \|\phi\|_0$ for $p \geq 0$, let (E_{-p}) denote the completion of (L^2) with respect to the norm $\|\cdot\|_{-p}$. Then for any $p \geq q$, $(E_p) \subset (E_q)$. It is easy to notice that (E_{-p}) is the dual space $(E_p)^*$ of (E_p). Let $(\mathcal{E})$ be the projective limit of the spaces $\{(E_p); p \geq 0\}$ and $(\mathcal{E})^*$ be the dual space of $(\mathcal{E})$. Then $(\mathcal{E}) = \cap_{p\geq 0} E_p$ and $(\mathcal{E})^* = \cup_{p\geq 0} E_{-p}$. Moreover, for any positive integer p, the inclusion map from (E_{p+q}) into (E_p) is a Hilbert-Schmidt operator if $q \geq 1$ (see, e.g., [KT 80a], [Kuo 92], [HKPS]). Thus, $(\mathcal{E})$ is a nuclear space and there is a Gel'fand triple:

$$(\mathcal{E}) \subset (L^2) \subset (\mathcal{E})^*.$$

$(\mathcal{E})$ is the space of Hida test functionals and $(\mathcal{E})^*$ is the space of Hida distributions. The characterization theorems for a particular choice of this Gel'fand triple,

$$(\mathcal{S}) \subset (L^2) \subset (\mathcal{S})^*$$

will be reviewed in the next section.

In the following, another Gel'fand triple is constructed:

$$(\mathcal{E}^\beta) \subset (L^2) \subset (\mathcal{E}^\beta)^*, \qquad \beta \in [0,1)$$

by generalizing a Gel'fand triple

$$(S^\beta) \subset L^2(S^*(I\!R), \mu) \subset (S^\beta)^*, \qquad \beta \in [0,1)$$

which has been introduced by Kontratiev and Streit in the paper [KS 92].

A number β in [0,1) is fixed and let $I\!N_0$ denote the set of all nonnegative integers. A family of norms $\{\|\cdot\|_{p,\beta}; p \in I\!N_0\}$ on (L^2) is defined by : for each $p \in I\!N_0$ and for $\phi = \sum_{n=0}^{\infty} I_n(f_n)$

$$\|\phi\|_{p,\beta}^2 \equiv \sum_{n=0}^{\infty} (n!)^{1+\beta} |f_n|_p^2$$

where $|f_n|_p = |(A^p)^{\otimes n} f_n|_0$. Let $(E_{p,\beta})$ be defined by

$$(E_{p,\beta}) \equiv \{\phi = \sum_{n=0}^{\infty} I_n(f_n) \in (L^2)\, ;\, \| \phi \|_{p,\beta} < \infty\}.$$

Another family of norms $\{\|\cdot\|_{-p,-\beta}; p \in I\!N_0\}$ on (L^2) is defined by: for each $p \in I\!N_0$ and for $\phi = \sum_{n=0}^{\infty} I_n(f_n)$

$$\|\phi\|_{-p,-\beta}^2 \equiv \sum_{n=0}^{\infty} (n!)^{1-\beta} |f_n|_{-p}^2$$

where $|f_n|_{-p} = |(A^{-p})^{\otimes n} f_n|_0$. Let $(E_{-p,-\beta})$ be the completion of (L^2) with respect to the norm $\|\cdot\|_{-p,-\beta}$. $(E_{-p,-\beta})$ can be identified as the dual space $(E_{p,\beta})^*$ of $(E_{p,\beta})$. For $p, q \in I\!N_0$, fixed $\beta \in [0,1)$, $(E_{q,\beta}) \subset (E_{p,\beta})$ if $q > p$. Let $(\mathcal{E}^\beta)$ denote the projective limit of the family of spaces $\{(E_{p,\beta}), p \in I\!N_0\}$. Then $(\mathcal{E}^\beta) = \cap_{p \geq 0} (E_{p,\beta})$. $(\mathcal{E}^{-\beta})$ is defined as the inductive limit of the family of spaces $\{(E_{-p,-\beta}), p \in I\!N_0\}$.

Then $(\mathcal{E}^{-\beta}) = \cup_{p\in N_0}(E_{-p,-\beta})$. Moreover, the space $(\mathcal{E}^{-\beta})$ can be identified as the dual space $(\mathcal{E}^{\beta})^*$ of $(\mathcal{E}^{\beta})$. Obviously, for all $\beta \in [0,1)$

$$(\mathcal{E}^{\beta}) \subset (\mathcal{E}), \quad (\mathcal{E})^* \subset (\mathcal{E}^{-\beta}).$$

In particular, $(\mathcal{E}^0) = (\mathcal{E})$ and $(\mathcal{E}^{-0}) = (\mathcal{E})^*$. Moreover, for any $p \in I\!N_0$ and any $\beta \in [0,1)$ the inclusion map from $(E_{p+1,\beta})$ into $(E_{p,\beta})$ is a Hilbert-Schmidt operator [KS 92]. Therefore, the space $(\mathcal{E}^{\beta})$, $\beta \in [0,1)$ is a nuclear space, and we have another Gel'fand triple

$$(\mathcal{E}^{\beta}) \subset (L^2) \subset (\mathcal{E}^{\beta})^* \qquad \beta \in [0,1).$$

There are inclusions:

$$(\mathcal{E}^{\beta}) \subset (\mathcal{E}) \subset (L^2) \subset (\mathcal{E})^* \subset (\mathcal{E}^{\beta})^*.$$

§1.2. The Characterizations of Hida Distributions

This section briefly describes the S-transforms on $(\mathcal{E})^*$, and $(\mathcal{E}^{\beta})^*$ which are useful tools in the study of the spaces $(\mathcal{E})^*$, $(\mathcal{E}^{\beta})^*$ and reviews the characterization theorems for the spaces $(\mathcal{S})^*, (\mathcal{S}), (\mathcal{S}^{\beta})^*$, and $(\mathcal{S})^{\beta}$.

For each $\xi \in \mathcal{E}$ the funcion $\exp(\langle\cdot,\xi\rangle - 2^{-1}|\xi|_0^2)$ belongs to the space $(\mathcal{E})$ [Kuo 92] and also $(\mathcal{E}^{\beta})$, $\beta \in [0,1)$[KS 92]. Let $\Phi \in (\mathcal{E})^*$and $\Psi \in (\mathcal{E}^{\beta})^*$. The S-transform of Φ is given by

$$(S\Phi)(\xi) = \langle\langle \Phi, \exp(\langle\cdot,\xi\rangle - 2^{-1}|\xi|_0^2)\rangle\rangle, \qquad \xi \in \mathcal{E},$$

where $\langle\langle\cdot,\cdot\rangle\rangle$ denotes the bilinear pairing between $(\mathcal{E})^*$ and$(\mathcal{E})$. The S-trasform of Ψ is in the same form

$$(S\Psi)(\xi) = \langle\langle \Psi, \exp(\langle\cdot,\xi\rangle - 2^{-1}|\xi|_0^2)\rangle\rangle, \qquad \xi \in \mathcal{E},$$

where we use $\langle\langle\cdot,\cdot\rangle\rangle$ to denote the bilinear pairing between $(\mathcal{E}^\beta)^*$ and$(\mathcal{E}^\beta)$. The S-transforms have been used in characterizing the spaces $(\mathcal{S})^*$ ([PS 91]), $(\mathcal{S})$ ([KPS 91]), $(\mathcal{S}^\beta)$, and $(\mathcal{S}^\beta)^*$([KS 92]). The characterization theorems for the spaces $(\mathcal{S})^*$ and $(\mathcal{S})$ have been reformulated in the paper [Kuo 92] as in the following Theorem 1.2.1 and Theorem 1.2.2. We reformulate the characterization theorems shown by Kontratiev and Streit for the spaces $(\mathcal{S}^\beta), (\mathcal{S}^\beta)^*$ in the following Theorem 1.2.3 and 1.2.4. For the proofs of those theorems refer to the corresponding papers as mentioned above.

Theorem 1.2.1.[PS 91]

The function F on $\mathcal{S}(\mathbb{R})$ is the S-transform of a Hida distribution in $(\mathcal{S})^*$ if and only if the fuction

$$f(\lambda,\xi) \equiv F(\lambda\xi), \quad \lambda \in \mathbb{R},\ \xi \in \mathcal{S}(\mathbb{R})$$

satisfies the following conditions;

(a) For any ξ and η in $\mathcal{S}(\mathbb{R})$, the function $F(\lambda\xi+\eta), \lambda \in \mathbb{R}$ extends to an entire function $F(z\xi+\eta), z \in \mathbb{C}$.

(b) There exist constants $p \geq 0, c \geq 0$, and $K \geq 0$ such that for all $z \in \mathbb{C}, \xi \in \mathcal{S}(\mathbb{R})$,

$$|f(z,\xi)| \leq K\exp(c|z|^2|\xi|_p^2).$$

Theorem 1.2.2. [KPS 91]

The function F on $\mathcal{S}(\mathbb{R})$ is the S-transform of a test functional in $(\mathcal{S})$ if and only if the function

$$f(\lambda,\xi) \equiv F(\lambda\xi), \quad \lambda \in \mathbb{R},\ \xi \in \mathcal{S}(\mathbb{R})$$

satisfies the following conditions;

(a) For each ξ and η in $\mathcal{S}(\mathbb{R})$, the function $F(\lambda\xi+\eta), \lambda \in \mathbb{R}$ extends to an entire function $F(z\xi+\eta), z \in \mathbb{C}$.

(b) For all $p \geq 0, \epsilon > 0$, there exists $K > 0$ such that for all $z \in \mathbb{C}, \xi \in \mathcal{S}(\mathbb{R})$,

$$|f(z,\xi)| \leq K\exp(\epsilon|z|^2|\xi|_{-p}^2).$$

Theorem 1.2.3. [KS 92]

The function F on $\mathcal{S}(\mathbb{R})$ is the S-transform of a test functional in $(\mathcal{S}^\beta)^*$ if and only if the function

$$f(\lambda,\xi) \equiv F(\lambda\xi), \quad \lambda \in \mathbb{R},\ \xi \in \mathcal{S}(\mathbb{R})$$

satisfies the following conditions;

(a) For each ξ in $\mathcal{S}(\mathbb{R})$, the function $F(\lambda\xi+\eta), \lambda \in \mathbb{R}$ extends to an entire function $F(z\xi+\eta), z \in \mathbb{C}$.

(b) There exist constants $p \geq 0, \epsilon > 0$ and $K > 0$ such that for all $z \in \mathbb{C}, \xi \in \mathcal{S}(\mathbb{R})$,

$$|f(z,\xi)| \leq K\exp(\epsilon|z|^2|\xi|_p^{\frac{2}{1-\beta}}).$$

Theorem 1.2.4. [KS 92]

The function F on $\mathcal{S}(\mathbb{R})$ is the S-transform of a white noise test functional in $(\mathcal{S}^\beta)$ if and only if the function

$$f(\lambda,\xi) \equiv F(\lambda\xi), \quad \lambda \in \mathbb{R},\ \xi \in \mathcal{S}(\mathbb{R})$$

satisfies the following conditions;

(a) For each ξ and η in $\mathcal{S}(\mathbb{R})$, the function $F(\lambda\xi+\eta), \lambda \in \mathbb{R}$ extends to an entire function $F(z\xi+\eta), z \in \mathbb{C}$.

(b) For all $p \geq 0, c > 0$, there exists $K > 0$ such that for all $z \in \mathbb{C}, \xi \in \mathcal{S}(\mathbb{R})$,

$$|f(z,\xi)| \leq K\exp(c|z|^2|\xi|_{-p}^{\frac{2}{1+\beta}}).$$

Chapter 2. Finite Dimensional Hida Distributions

This chapter reviews briefly some of the results from the paper [KK 92] which are necessary for this dissertation, in particular, the characterization theorems for the spaces $\mathcal{F}(\mathbb{R}^k), \mathcal{F}^*(\mathbb{R}^k)$ and the space of finite dimensional Hida distributions. we refer to the paper [KK 92] for their proofs.

For consistency, the same notation is used as in the paper [KK 92] for multi-indices as follows: For $\mathbf{n} = (n_1, \cdots, n_k) \in \mathbb{N}_0^k$, $\mathbf{m} = (m_1, \cdots, m_k) \in \mathbb{N}_0^k$, $\mathbf{u} = (u_1, \cdots, u_k) \in \mathbb{R}^k$, $\mathbf{v} = (v_1, \cdots, v_k) \in \mathbb{R}^k$, and $\mathbf{z} = (z_1, \cdots, z_k) \in \mathbb{C}^k$, define

$$\mathbf{n}! \equiv n_1! \cdots n_k!\,,$$

$$|\mathbf{n}| \equiv n_1 + \cdots + n_k\,,$$

$$\mathbf{m} \leq \mathbf{n} \text{ means that } m_j \leq n_j \text{ for all } 1 \leq j \leq k\,,$$

$$a_{\mathbf{n}} \equiv a_{n_1, \ldots, n_k}\,,$$

$$\mathbf{u}^{\mathbf{n}} \equiv u_1^{n_1} \cdots u_k^{n_k}\,,$$

$$\mathbf{u}^2 \equiv u_1^2 + \cdots + u_k^2\,,$$

$$\mathbf{uv} \equiv u_1 v_1 + \cdots + u_k v_k\,,$$

$$|\mathbf{z}|^2 \equiv |z_1|^2 + \cdots + |z_k|^2\,,$$

$$(\partial/\partial \mathbf{u})^{\mathbf{n}} \equiv \partial^{|\mathbf{n}|}/\partial u_1^{n_1} \cdots \partial u_k^{n_k}.$$

§2.1. The Spaces $\mathcal{H}(\mathbb{R}^k), \mathcal{H}^*(\mathbb{R}^k), \mathcal{F}(\mathbb{R}^k)$, and $\mathcal{F}^*(\mathbb{R}^k)$

From the Schwartz distribution theory, recall the Gel'fand triple

$$\mathcal{S}(\mathbb{R}^k) \subset L^2(\mathbb{R}^k) \subset \mathcal{S}^*(\mathbb{R}^k)$$

which is associated with the Lebesgue measure on $\mathbb{R}^k$. A new Gel'fand triple over

$\mathbb{R}^k$ which is associated with the standard Gaussian measure μ_k on $\mathbb{R}^k$ has been introduced by Kubo and Kuo [KK 92]. This Gel'fand triple has played a key role for their work in the paper [KK 92]. The following describes the new Gel'fand triple $\mathcal{H}(\mathbb{R}^k) \subset \mathcal{H}_0(\mathbb{R}^k) \subset \mathcal{H}^*(\mathbb{R}^k)$ as constructed in the paper [KK 92].

Let $\mathcal{P}(\mathbb{R}^k)$ be the vector space of polynomials in $\mathbf{u} \in \mathbb{R}^k$. Define an inner product $(\cdot,\cdot)_{\mathcal{H}}$ on $\mathcal{P}(\mathbb{R}^k)$ by

$$(F,G)_{\mathcal{H}} \equiv \int_{\mathbb{R}^k} F(\mathbf{u})\overline{G(\mathbf{u})}\, d\mu_k(\mathbf{u})$$

where $F(\mathbf{u})$, $G(\mathbf{u}) \in \mathcal{P}(\mathbb{R}^k)$. In order to construct a new Gel'fand triple over $\mathbb{R}^k$, a family $\{(\cdot,\cdot)_{\mathcal{H},t}; t \in \mathbb{R}\}$ of inner products on $\mathcal{P}(\mathbb{R}^k)$ is introduced. Denote L to be the Ornstein-Uhlenbeck operator on $\mathbb{R}^k$, i.e.,

$$L \equiv \Delta - u_1\frac{\partial}{\partial u_1} - \cdots - u_k\frac{\partial}{\partial u_k}$$

where Δ is the Laplacian operator on $\mathbb{R}^k$

$$\Delta \equiv \frac{\partial^2}{\partial u_1^2} + \cdots + \frac{\partial^2}{\partial u_k^2}.$$

Then for each $t \in \mathbb{R}$, define an inner product $(\cdot,\cdot)_{\mathcal{H},t}$ on $\mathcal{P}(\mathbb{R}^k)$ by

$$\begin{aligned}(F,G)_{\mathcal{H},t} &\equiv (e^{-tL}F, e^{-tL}G)_{\mathcal{H}}\\ &= \int_{\mathbb{R}^k} (e^{-tL}F)(\mathbf{u})\overline{(e^{-tL}G)(\mathbf{u})}\, d\mu_k(\mathbf{u}).\end{aligned}$$

Let $\|\cdot\|_{\mathcal{H},t}$ be the norm on $\mathcal{P}(\mathbb{R}^k)$ corresponding to the inner product $(\cdot,\cdot)_{\mathcal{H},t}$ for each $t \in \mathbb{R}$. Then for $F \in \mathcal{P}(\mathbb{R}^k)$,

$$\|F\|_{\mathcal{H},t} \equiv \|e^{-tL}F\|_{\mathcal{H}}.$$

For each $t \in \mathbb{R}$, define the space $\mathcal{H}_t(\mathbb{R}^k)$ to be the completion of $\mathcal{P}(\mathbb{R}^k)$ with respect to the norm $\|\cdot\|_{\mathcal{H},t}$. Let $H_n(u)$ be the Hermite polynomial of degree n on

$I\!R$ defined by the generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(u) = \exp[tu - \frac{1}{2}t^2]$$

and $H_{\mathbf{n}}(\mathbf{u})$ be the Hermite polynomial with multi indices given by

$$H_{\mathbf{n}}(\mathbf{u}) \equiv H_{n_1}(u_1) H_{n_2}(u_2) \cdots H_{n_k}(u_k).$$

By using the following equations,

$$\left(\frac{d^2}{du^2} - u\frac{d}{du}\right) H_n(u) = -nH_n(u),$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} H_n(u) H_m(u) e^{-\frac{1}{2}u^2}\, du = \delta_{n,m} n!,$$

it is easy to see that $e^{-tL} H_{\mathbf{n}} = e^{|\mathbf{n}|t} H_{\mathbf{n}}$ for any $t \in I\!R$ and

$$(H_{\mathbf{n}}, H_{\mathbf{m}})_{\mathcal{H},t} = \delta_{\mathbf{n},\mathbf{m}} \mathbf{n}! e^{2|\mathbf{n}|t}.$$

This shows that $\{H_{\mathbf{n}} : \mathbf{n} \geq 0\}$ is a complete orthogonal system in each $\mathcal{H}_t(I\!R^k)$ with respect to the inner product $(\cdot,\cdot)_{\mathcal{H},t}$. Thus any element F in $\mathcal{H}_t(I\!R^k)$ can be represented by a linear combination of Hermite polynomials $H_{\mathbf{n}}$, i.e., $F = \sum a_{\mathbf{n}} H_{\mathbf{n}}$. For $F = \sum a_{\mathbf{n}} H_{\mathbf{n}}$ and $G = \sum b_{\mathbf{n}} H_{\mathbf{n}}$ in $\mathcal{H}_t(I\!R^k)$ we have

$$(F, G)_{\mathcal{H},t} = \sum \mathbf{n}! e^{2|\mathbf{n}|t} a_{\mathbf{n}} \overline{b_{\mathbf{n}}},$$

$$\|F\|^2_{\mathcal{H},t} = \sum \mathbf{n}! e^{2|\mathbf{n}|t} |a_{\mathbf{n}}|^2.$$

The clear fact that $\|\cdot\|_{\mathcal{H},s} \leq \|\cdot\|_{\mathcal{H},t}$ for any $t \geq s$ implies that for any $t \geq s$

$$\mathcal{H}_t(I\!R^k) \subset \mathcal{H}_s(I\!R^k).$$

$\mathcal{H}_{-t}(I\!R^k)$, $t \geq 0$ can be identified as the dual space $\mathcal{H}_t^*(I\!R^k)$ of $\mathcal{H}_t(I\!R^k)$ with the bilinear pairing $\langle\cdot,\cdot\rangle$ given by: for $F = \sum a_{\mathbf{n}} H_{\mathbf{n}} \in \mathcal{H}_{-t}(I\!R^k)$ and $G = \sum b_{\mathbf{n}} H_{\mathbf{n}} \in \mathcal{H}_t(I\!R^k)$

$$\langle F, G\rangle \equiv \sum \mathbf{n}! a_{\mathbf{n}} b_{\mathbf{n}}.$$

For any $t \geq 0$ there are the following inclusions:

$$\mathcal{H}_t(\mathbb{R}^k) \subset \mathcal{H}_0(\mathbb{R}^k) \subset \mathcal{H}_t^*(\mathbb{R}^k).$$

The convention between $\langle \cdot, \cdot \rangle$ and the inner product $(\cdot, \cdot)_{\mathcal{H}}$ has been used, i.e.,

$$\langle F, G \rangle = (F, \overline{G})_{\mathcal{H},0}, \quad F \in \mathcal{H}_0,\ G \in \mathcal{H}_t.$$

Now, define $\mathcal{H}(\mathbb{R}^k)$ to be the projective limit of the spaces $\{\mathcal{H}_t(\mathbb{R}^k)\,; t \in \mathbb{R}\}$ and $\mathcal{H}^*(\mathbb{R}^k)$ to be the dual space of $\mathcal{H}(\mathbb{R}^k)$. Then $\mathcal{H}(\mathbb{R}^k) = \cap_t \mathcal{H}_t(\mathbb{R}^k)$ and $\mathcal{H}^*(\mathbb{R}^k) = \cup_t \mathcal{H}_t(\mathbb{R}^k)$. Moreover, the inclusion map i from $\mathcal{H}_t(\mathbb{R}^k)$ into $\mathcal{H}_s(\mathbb{R}^k)$, $t > s$, is a Hilbert-Schmidt operator. Therefore, there is the new Gel'fand triple

$$\mathcal{H}(\mathbb{R}^k) \subset \mathcal{H}_0(\mathbb{R}^k) \subset \mathcal{H}^*(\mathbb{R}^k).$$

Note here that $\mathcal{H}_0(\mathbb{R}^k) = L^2(\mathbb{R}^k, \mu_k)$, the space of square integrable functions with respect to the standard Gaussian measure μ_k.

In the following, some properties of the spaces $\mathcal{H}_t(\mathbb{R}^k), \mathcal{H}(\mathbb{R}^k)$ and $\mathcal{H}^*(\mathbb{R}^k)$, in particular, the relationship between the Gel'fand triples $\mathcal{S}(\mathbb{R}^k) \subset L^2(\mathbb{R}^k) \subset \mathcal{S}^*(\mathbb{R}^k)$ and $\mathcal{H}(\mathbb{R}^k) \subset \mathcal{H}_0(\mathbb{R}^k) \subset \mathcal{H}^*(\mathbb{R}^k)$ will be reviewed.

Theorem 2.1.1.[KK 92]

Every function F in $\mathcal{H}_t(\mathbb{R}^k), t \geq 0$, is smooth and satisfies the following inequality

$$|F(\mathbf{u})| \leq (1.2)^k (1 - e^{-2t})^{-k/2} \|F\|_{\mathcal{H},t} \exp[\mathbf{u}^2/4], \quad \mathbf{u} \in \mathbb{R}^k.$$

Theorem 2.1.2.[KK 92]

The partial derivatives $\frac{\partial}{\partial u_i}$, $1 \le i \le k$, are continuous from $\mathcal{H}(\mathbb{R}^k)$ into itself and also from $\mathcal{H}^*(\mathbb{R}^k)$ into itself.

Theorem 2.1.3.[KK 92]

The following assertions hold:

(1) If $F \in \mathcal{H}(\mathbb{R}^k)$, then $Fe^{-\frac{1}{4}\mathbf{u}^2} \in \mathcal{S}(\mathbb{R}^k)$, i.e.,

$$\{Fe^{-\frac{1}{4}\mathbf{u}^2}; F \in \mathcal{H}(\mathbb{R}^k)\} \subset \mathcal{S}(\mathbb{R}^k);$$

(2) If $F \in \mathcal{S}^*(\mathbb{R}^k)$, then $Fe^{\frac{1}{4}\mathbf{u}^2} \in \mathcal{H}^*(\mathbb{R}^k)$, i.e.,

$$\mathcal{S}^*(\mathbb{R}^k) \subset \{Ge^{-\frac{1}{4}\mathbf{u}^2}; G \in \mathcal{H}^*(\mathbb{R}^k)\};$$

(3) $\mathcal{S}^*(\mathbb{R}^k) \subset \mathcal{H}^*(\mathbb{R}^k)$.

Note that there is no inclusion relationship between $\mathcal{H}(\mathbb{R}^k)$ and $\mathcal{S}(\mathbb{R}^k)$. For example, it is easy to check from the direct computation of the norm that $e^u \in \mathcal{H}(\mathbb{R}^k)$, but $e^u \notin \mathcal{S}(\mathbb{R}^k)$. On the other hand $e^{-u^2} \in \mathcal{S}(\mathbb{R}^k)$, but it is also easy to check from the following characterization theorem for the space $\mathcal{H}(\mathbb{R}^k)$ (see Theorem 2.1.6, 2.1.8) that $e^{-u^2} \notin \mathcal{H}(\mathbb{R}^k)$.

Now, for characterizations of the spaces $\mathcal{H}(\mathbb{R}^k)$ and $\mathcal{H}^*(\mathbb{R}^k)$ another new Gel'fand triple $\mathcal{F}(\mathbb{R}^k) \subset \mathcal{F}_0(\mathbb{R}^k) \subset \mathcal{F}^*(\mathbb{R}^k)$ and a transform from $\mathcal{H}^*(\mathbb{R}^k)$ into $\mathcal{F}^*(\mathbb{R}^k)$ have been introduced. The following decribes briefly a Gel'fand triple $\mathcal{F}(\mathbb{R}^k) \subset \mathcal{F}_0(\mathbb{R}^k) \subset \mathcal{F}^*(\mathbb{R}^k)$ and the σ-transform.

Let $F(\mathbf{u}) = \sum a_{\mathbf{n}}\mathbf{u}^{\mathbf{n}}$ and $G(\mathbf{u}) = \sum b_{\mathbf{n}}\mathbf{u}^{\mathbf{n}}$ in $\mathcal{P}(\mathbb{R}^k)$. Define another inner product $(\cdot,\cdot)_{\mathcal{F}}$ on $\mathcal{P}(\mathbb{R}^k)$ by

$$(F,G)_{\mathcal{F}} \equiv \sum_{\mathbf{n}} \mathbf{n}! a_{\mathbf{n}} \overline{b_{\mathbf{n}}}.$$

For construction of another Gel'fand triple, define a family of an inner products $\{(\cdot,\cdot)_{\mathcal{F},t},\ t \in \mathbb{R}\}$ on $\mathcal{P}(\mathbb{R}^k)$ by

$$(F, G)_{\mathcal{F},t} \equiv \sum \mathbf{n}! e^{2|\mathbf{n}|t} a_{\mathbf{n}} \overline{b_{\mathbf{n}}}, \qquad t \in \mathbb{R}.$$

Let $\|\cdot\|_{\mathcal{F},t}$ denote the norm corresponding to the inner product $(\cdot,\cdot)_{\mathcal{F},t}$. Then

$$\|F\|^2_{\mathcal{F},t} \equiv \sum \mathbf{n}! e^{2|\mathbf{n}|t} |a_{\mathbf{n}}|^2 .$$

Now, denote $\mathcal{F}_t(\mathbb{R}^k)$ to be the completion of $\mathcal{P}(\mathbb{R}^k)$ with respect to the norm $\|\cdot\|_{\mathcal{F},t}$ for each $t \in \mathbb{R}$. It is clear that the collection $\{\mathbf{u}^{\mathbf{n}};\ \mathbf{n} \geq 0\}$ is a complete orthogonal system in each $\mathcal{F}_t(\mathbb{R}^k)$ with respect to the inner product $(\cdot,\cdot)_{\mathcal{F},t}$. It is obvious that for any $t \geq s$,

$$\mathcal{F}_t(\mathbb{R}^k) \subset \mathcal{F}_s(\mathbb{R}^k).$$

For $t \geq 0$ the space $\mathcal{F}_{-t}(\mathbb{R}^k)$ can be identified as the dual space $\mathcal{F}_t^*(\mathbb{R}^k)$ of $\mathcal{F}_t(\mathbb{R}^k)$ with respect to the bilinear pairing $\langle\cdot,\cdot\rangle$. For $F = \sum a_{\mathbf{n}} \mathbf{u}_{\mathbf{n}} \in \mathcal{F}_{-t}(\mathbb{R}^k)$ and $G = \sum b_{\mathbf{n}} \mathbf{u}_{\mathbf{n}} \in \mathcal{F}_t(\mathbb{R}^k)$

$$\langle F, G\rangle \equiv \sum \mathbf{n}! a_{\mathbf{n}} b_{\mathbf{n}}.$$

Define the space $\mathcal{F}(\mathbb{R}^k)$ as the projective limit of the spaces $\{\mathcal{F}_t(\mathbb{R}^k); t \in \mathbb{R}\}$ and $\mathcal{F}^*(\mathbb{R}^k)$ as the dual space of $\mathcal{F}(\mathbb{R}^k)$. Then $\mathcal{F}(\mathbb{R}^k) = \cap_t \mathcal{F}_t(\mathbb{R}^k)$ and $\mathcal{F}^*(\mathbb{R}^k) = \cup_t \mathcal{F}_t(\mathbb{R}^k)$. Furthermore, the inclusion map i from $\mathcal{F}_t(\mathbb{R}^k)$ into $\mathcal{F}_s(\mathbb{R}^k)$, $t > s$, is a Hilbert-Schmidt operator. Therefore, we have another new Gel'fand triple over $\mathbb{R}^k$

$$\mathcal{F}(\mathbb{R}^k) \subset \mathcal{F}_0(\mathbb{R}^k) \subset \mathcal{F}^*(\mathbb{R}^k).$$

For each $\mathbf{v} \in \mathbb{R}^k$, we let $e_{\mathbf{v}}(\mathbf{u}) = \exp(\mathbf{uv} - \mathbf{v}^2/2)$ for simplicity. We can check that for all $\mathbf{v} \in \mathbb{R}^k, e_{\mathbf{v}} \in \mathcal{H}(\mathbb{R}^k)$ by using the generating function for the Hermite polynomials.

The σ-transform of $F \in \mathcal{H}^*(\mathbb{R}^k)$ is defined by

$$(\sigma F)(\mathbf{v}) \equiv \langle F, e_{\mathbf{v}} \rangle, \quad \mathbf{v} \in \mathbb{R}^k.$$

Also, the σ-transform of $F \in \mathcal{H}^*(\mathbb{R}^k)$ can be formally expressed by using the relationship between the dual pairing $\langle \cdot, \cdot \rangle$, and inner product $(\cdot, \cdot)_{\mathcal{H}}$ as follows:

$$\sigma F(\mathbf{v}) = \int_{\mathbb{R}^k} F(\mathbf{u} + \mathbf{v}) d\mu_k(\mathbf{v}),$$

where μ_k is the standard Gaussian measure on $\mathbb{R}^k$. By using the above expression for the σ-transform and the generating function for the Hermite polynomials we can derive the following:

$$\begin{aligned}
(\sigma H_n)(v) &= \int_{\mathbb{R}} H_n(u+v)\, d\mu_1(u) \\
&= \int_{\mathbb{R}} H_n(x) \exp[xv - \frac{1}{2}v^2]\, d\mu_1(x) \\
&= \int_{\mathbb{R}} H_n(x) \left(\sum_{m=0}^{\infty} \frac{v^m}{m!} H_m(x) \right) d\mu_1(x) \\
&= v^n.
\end{aligned}$$

This implies that

$$(\sigma H_{\mathbf{n}})(\mathbf{v}) = \mathbf{v}^{\mathbf{n}}.$$

Therefore, if $F \in \mathcal{H}_t(\mathbb{R}^k), t \in \mathbb{R}$ is represented by $F(\mathbf{u}) = \sum a_{\mathbf{n}} H_{\mathbf{n}}(\mathbf{u})$, then

$$(\sigma F)(\mathbf{v}) = \sum a_{\mathbf{n}} \mathbf{v}^{\mathbf{n}}.$$

Moreover,

$$\|F\|_{\mathcal{H},t} = \|\sigma F\|_{\mathcal{F},t}.$$

From the representations of F and σF, it is clear that the $\sigma; \mathcal{H}_t(\mathbb{R}^k) \mapsto \mathcal{F}_t(\mathbb{R}^k)$ is onto for each $t \in \mathbb{R}$. Thus, we have the following:

Theorem 2.1.4. [KK 92]

The σ-transform is an isometry from $\mathcal{H}_t(\mathbb{R}^k)$ onto $\mathcal{F}_t(\mathbb{R}^k)$ for each $t \in \mathbb{R}$.

Corollary 2.1.5. [KK 92]

The σ-transform is homeomorphic from $\mathcal{H}(\mathbb{R}^k)$ onto $\mathcal{F}(\mathbb{R}^k)$ and also from $\mathcal{H}^*(\mathbb{R}^k)$ onto $\mathcal{F}^*(\mathbb{R}^k)$.

The following reviews the characterization theorems for the spaces $\mathcal{F}_t(\mathbb{R}^k)$, $\mathcal{F}_t^*(\mathbb{R}^k)$. We refer to the paper [KK 92] for their proofs.

Theorem 2.1.6. [KK 92]

A function F defined on $\mathbb{R}^k$ is in $\mathcal{F}(\mathbb{R}^k)$ if and only if $F(\mathbf{v})$, $\mathbf{v} \in \mathbb{R}^k$, has an entire extension $F(\mathbf{z})$, $\mathbf{z} \in \mathbb{C}^k$, and satisfies the following condition: for any $a > 0$ there exists a constant $A > 0$ such that

$$|F(\mathbf{z})| \leq A \exp(a\,|\mathbf{z}|^2), \qquad \mathbf{z} \in \mathbb{C}^k.$$

Theorem 2.1.7. [KK 92]

A function F defined on $\mathbb{R}^k$ is in $\mathcal{F}^*(\mathbb{R}^k)$ if and only if $F(\mathbf{v})$, $\mathbf{v} \in \mathbb{R}^k$, has an entire extension $F(\mathbf{z})$, $\mathbf{z} \in \mathbb{C}^k$, and there exist constants $b > 0$ and $A > 0$ such that

$$|F(\mathbf{z})| \leq A \exp(b\,|\mathbf{z}|^2),\ \mathbf{z} \in \mathbb{C}^k.$$

Theorem 2.1.8. [KK 92]

$\mathcal{F}(\mathbb{R}^k) = \mathcal{H}(\mathbb{R}^k)$ as sets.

§2.2. Finite Dimensional Hida Distributions

This section briefly describes the finite dimensional Hida distributions and their characterization theorems in [KK 92].

Recall the Gel'fand triple $\mathcal{E} \subset E \subset \mathcal{E}^*$ from §1.1. For each η in $\mathcal{E}$ the random variable $\langle \cdot, \eta \rangle$ is defined everywhere on $\mathcal{E}^*$ and $\langle \cdot, \eta \rangle$ has normal distribution with mean 0 and variance $|\eta|_0^2$. For a funcion $f \in E$, choose a sequence $\{\eta_n\}$ in $\mathcal{E}$ such that the sequence η_n is Cauchy in (L^2) and hence converges to f in E. Then it is well known that the sequence $\{\langle \cdot, \eta_n \rangle\}$ converges to an element in (L^2) which is denoted by $\langle \cdot, f \rangle$. Note that $\langle \cdot; f \rangle$ is independent of the choice of sequence $\{\eta_n\}$ and $\langle \cdot, f \rangle$ has normal distribution with mean zero and variance $|f|_0^2$.

Let $\{e_1, \cdots, e_k\} \subset E$ be a finite collection of linearly independent elements in E and denote V to be the linear space spanned by the collection $\{e_1, \cdots, e_k\}$ by setting $V \equiv \operatorname{span}\{e_1, \cdots, e_k\}$. For simplicity, set $\langle \cdot, \vec{e} \rangle \equiv (\langle \cdot, e_1 \rangle, \cdots, \langle \cdot, e_k \rangle)$. Polynomials in $\langle \cdot, \vec{e} \rangle$ are elements in (L^2).

Definition 2.2.1. [KK 92]

Let $(\mathcal{E})_V^* \equiv (\mathcal{E})^*$-closure of all polynomials in $\langle \cdot, \vec{e} \rangle$ for some finite dimensional subspace V spanned by $\{e_1, \cdots, e_k\} \subset E$. Elements in $(\mathcal{E})_V^*$ are called finite dimensional Hida distributions. In this case, we say that $\Phi \in (\mathcal{E})_V^*$ is based on V.

In the following Theorem 2.2.2 and Theorem 2.2.3 the space of finite dimensional Hida distributions are characterized, and we refer to the paper [KK 92] for their proofs. These two theorems characterize such fuctions F that $F \circ (\langle \cdot, \xi_1 \rangle, \cdots, \langle \cdot, \xi_k \rangle) \in (\mathcal{E})^*$, $\xi \in E$ at the same time. In other words, the set of finite dimensional Hida distributions coincides with the set of all compositions of generalized functions in $\mathcal{H}^*(\mathbb{R}^k)$ with Gaussian random variables.

Theorem 2.2.2. [KK 92]

Let $\{e_1, \cdots, e_k\}$ be orthonormal in E and let $V \equiv \text{span}\{e_1, \cdots, e_k\}$.

(1) If $F = \sum_{\mathbf{n}} a_{\mathbf{n}} H_{\mathbf{n}} \in \mathcal{H}^*(\mathbb{R}^k)$, then $\sum_{|\mathbf{n}|=0}^{N} a_{\mathbf{n}} H_{\mathbf{n}}(\langle \cdot, \vec{e} \rangle)$ converges to an element in $(\mathcal{E})^*$ as N goes to infinity;

(2) Let

$$F(\langle \cdot, \vec{e} \rangle) \equiv \lim_{N \to \infty} \sum_{|\mathbf{n}|=0}^{N} a_{\mathbf{n}} H_{\mathbf{n}}(\langle \cdot, \vec{e} \rangle).$$

Then $F(\langle \cdot, \vec{e} \rangle) \in (\mathcal{E})_V^*$ with S-transform given by

$$SF(\langle \cdot, \vec{e} \rangle)(\xi) = (\sigma F)(\langle \xi, \vec{e} \rangle), \qquad \xi \in \mathcal{E},$$

where σF is the σ-transform of F;

(3) The mapping $F \mapsto F(\langle \cdot, \vec{e} \rangle)$ from $\mathcal{H}^*(\mathbb{R}^k)$ into $(\mathcal{E})^*$ is continuous.

Theorem 2.2.3. [KK 92]

Let $\{e_1, \cdots, e_k\}$ be orthonormal in E and let $V \equiv \text{span}\{e_1, \cdots, e_k\}$. Suppose that $\Phi \in (\mathcal{E})_V^*$. Then there exists a function $F \in \mathcal{H}^*(\mathbb{R}^k)$ such that

$$\Phi = F(\langle \cdot, e_1 \rangle, \cdots, \langle \cdot, e_k \rangle).$$

In the following Theorem 2.2.4 and Theorem 2.2.5 the finite dimensional Hida distributions belonging to the space $(\mathcal{E})$ of test funtionals are characterized, and we also refer to the paper [KK 92] for their proofs. These two theorems characterize such fuctions F that $F \circ (\langle \cdot, \xi_1 \rangle, \cdots, \langle \cdot, \xi_k \rangle) \in (\mathcal{E})$, $\xi \in \mathcal{E}$. In other words, the set $(\mathcal{E})_V$ coincides with the set of all compositions of functions in $\mathcal{H}(\mathbb{R}^k)$ with Gaussian random variables.

Theorem 2.2.4. [KK 92]

Let $\{\eta_1, \cdots, \eta_k\} \subset \mathcal{E}$ be orthonormal in E and let $V \equiv \text{span}\{\eta_1, \cdots, \eta_k\}$. If $F \in \mathcal{H}(\mathbb{R}^k)$, then $F(\langle \cdot, \vec{\eta} \rangle) \in (\mathcal{E})_V \equiv (\mathcal{E}) \cap (\mathcal{E})^*_V$. Moreover, the mapping $F \mapsto F(\langle \cdot, \vec{\eta} \rangle)$ is continuous from $\mathcal{H}(\mathbb{R}^k)$ into $(\mathcal{E})$.

Theorem 2.2.5. [KK 92]

Let $\{\eta_1, \cdots, \eta_k\} \subset \mathcal{E}$ be orthonormal in E and let $V \equiv \text{span}\{\eta_1, \cdots, \eta_k\}$. If $\phi \in (\mathcal{E})_V$, then there exists a function $F \in \mathcal{H}(\mathbb{R}^k)$ such that

$$\phi = F(\langle \cdot, \eta_1 \rangle, \cdots, \langle \cdot, \eta_k \rangle).$$

Chapter 3. Finite Dimensional Hida Distributions of Order β

In this chapter, we will introduce the new spaces $\mathcal{H}^\beta(\mathbb{R}^k), (\mathcal{H}^\beta)^*(\mathbb{R}^k), \mathcal{F}^\beta(\mathbb{R}^k)$, and $(\mathcal{F}^\beta)^*(\mathbb{R}^k)$ and generalize the results mentioned in Chapter 2 to these spaces. The idea is based on [KK 92] and we will modify the arguments and the calculations in [KK 92] to obtain the corresponding results for $\mathcal{H}^\beta(\mathbb{R}^k), (\mathcal{H}^\beta)^*(\mathbb{R}^k), \mathcal{F}^\beta(\mathbb{R}^k)$, and $(\mathcal{F}^\beta)^*(\mathbb{R}^k)$.

§3.1. The Spaces $\mathcal{H}^\beta(\mathbb{R}^k), (\mathcal{H}^\beta)^*(\mathbb{R}^k), \mathcal{F}^\beta(\mathbb{R}^k)$, and $(\mathcal{F}^\beta)^*(\mathbb{R}^k)$

In the following, a new Gel'fand triple $\mathcal{H}^\beta(\mathbb{R}^k) \subset \mathcal{H}_0^0(\mathbb{R}^k) \subset (\mathcal{H}^\beta)^*(\mathbb{R}^k)$ which plays a key role for the object of this dissertation is introduced. This Gel'fand triple can be regarded as the finite dimensional version of $(\mathcal{E}^\beta) \subset (L^2) \subset (\mathcal{E}^\beta)^*$.

First define an operator W_α on $\mathcal{P}(\mathbb{R}^k)$ by

$$W_\alpha H_{\mathbf{n}}(\mathbf{u}) \equiv (\mathbf{n}!)^{\alpha/2} H_{\mathbf{n}}(\mathbf{u})$$

where $\alpha \in (-1, 1)$. Define an inner product $(\cdot,\cdot)_{\mathcal{H},\alpha}$ on $\mathcal{P}(\mathbb{R}^k)$ by

$$(F, G)_{\mathcal{H},\alpha} = \int_{R^k} (W_\alpha F)(\mathbf{u})\overline{(W_\alpha G)(\mathbf{u})}\, d\mu_k(\mathbf{u})$$

where $F(\mathbf{u})$, $G(\mathbf{u}) \in \mathcal{P}(\mathbb{R}^k)$. Fix $\beta \in [0, 1)$. In order to construct the Gel'fand triple mentioned above, we introduce a family $\{(\cdot,\cdot)_{\mathcal{H},\beta,t}; t \geq 0\}$ of inner products on $\mathcal{P}(\mathbb{R}^k)$. For each $t \geq 0$ define an inner product $(\cdot,\cdot)_{\mathcal{H},\beta,t}$ on $\mathcal{P}(\mathbb{R}^k)$ by

$$\begin{aligned}(F, G)_{\mathcal{H},\beta,t} &\equiv (e^{-tL}F, e^{-tL}G)_{\mathcal{H},\beta} \\ &= \int_{R^k} (W_\beta e^{-tL}F)(\mathbf{u})\overline{(W_\beta e^{-tL}G)(\mathbf{u})}\, d\mu_k(\mathbf{u}).\end{aligned}$$

For each $t \geq 0$, let $\|\cdot\|_{\mathcal{H},\beta,t}$ be the norm corresponding to the inner product $(\cdot,\cdot)_{\mathcal{H},\beta,t}$. Then

$$\|F\|^2_{\mathcal{H},\beta,t} \equiv (e^{-tL}F, e^{-tL}F)_{\mathcal{H},\beta}.$$

Denote $\mathcal{H}^{\beta}_{t}(\mathbb{R}^k)$ to be the completion of $\mathcal{P}(\mathbb{R}^k)$ with respect to the norm $\|\cdot\|_{\mathcal{H},\beta,t}$ and $\mathcal{H}^{-\beta}_{-t}(\mathbb{R}^k)$ be the completion of $\mathcal{P}(\mathbb{R}^k)$ with respect to the norm $\|\cdot\|_{\mathcal{H},-\beta,-t}$ defined on $\mathcal{P}(\mathbb{R}^k)$ by

$$\|F\|^2_{\mathcal{H},-\beta,-t} \equiv (e^{tL}F, e^{tL}F)_{\mathcal{H},-\beta}$$

for each $t \geq 0$. From recalling the equation $e^{-tL}H_{\mathbf{n}} = e^{|\mathbf{n}|t}H_{\mathbf{n}}$ for any $t \in \mathbb{R}$ we have the following

Fact 3.1.1.

For any $t \geq 0$,

$$(H_{\mathbf{n}}, H_{\mathbf{m}})_{\mathcal{H},\beta,t} = \delta_{\mathbf{n},\mathbf{m}}(\mathbf{n}!)^{1+\beta}e^{2|\mathbf{n}|t},$$

$$(H_{\mathbf{n}}, H_{\mathbf{m}})_{\mathcal{H},-\beta,-t} = \delta_{\mathbf{n},\mathbf{m}}(\mathbf{n}!)^{1-\beta}e^{-2|\mathbf{n}|t}.$$

Proof.

$$\begin{aligned}
(H_{\mathbf{n}}, H_{\mathbf{m}})_{\mathcal{H},\beta,t} &= \int_{\mathbb{R}^k} (W_\beta e^{-tL}H_{\mathbf{n}})(\mathbf{u})\overline{(W_\beta e^{-tL}H_{\mathbf{m}})(\mathbf{u})}\, d\mu_k(\mathbf{u}) \\
&= \int_{\mathbb{R}^k} (\mathbf{n}!)^{\beta/2}e^{|\mathbf{n}|t}H_{\mathbf{n}}(\mathbf{u})(\mathbf{m}!)^{\beta/2}e^{|\mathbf{m}|t}\overline{H_{\mathbf{m}}(\mathbf{u})}\, d\mu_k(\mathbf{u}) \\
&= \begin{cases} (\mathbf{n}!)^{\beta}e^{2|\mathbf{n}|t}\int_{\mathbb{R}^k}|H_{\mathbf{n}}(\mathbf{u})|^2 d\mu_k(\mathbf{u}) = (\mathbf{n}!)^{1+\beta}e^{2|\mathbf{n}|t} \text{ if } \mathbf{n}=\mathbf{m} \\ \\ (\mathbf{n}!\mathbf{m}!)^{\beta/2}e^{(|\mathbf{n}|+|\mathbf{m}|)t}\int_{\mathbb{R}^k} H_{\mathbf{n}}(\mathbf{u})\overline{H_{\mathbf{m}}(\mathbf{u})}\, d\mu_k(\mathbf{u}) = 0 \text{ if } \mathbf{n}\neq\mathbf{m} \end{cases} \\
&= \delta_{\mathbf{n},\mathbf{m}}(\mathbf{n}!)^{1+\beta}e^{2\mathbf{n}t}.
\end{aligned}$$

Similar calculation gives $(H_{\mathbf{n}}, H_{\mathbf{m}})_{\mathcal{H},-\beta,-t} = \delta_{\mathbf{n},\mathbf{m}}(\mathbf{n}!)^{1-\beta}e^{-2|\mathbf{n}|t}$. *Q.E.D.*

It follows from Fact 3.1.1 that $\{H_{\mathbf{n}}; \mathbf{n} \geq 0\}$ is a complete orthogonal system for each $\mathcal{H}_t^{\beta}(\mathbb{R}^k)$ with respect to the inner product $(\cdot,\cdot)_{\mathcal{H},\beta,t}$ and also for each $\mathcal{H}_{-t}^{-\beta}(\mathbb{R}^k)$ with respect to the inner product $(\cdot,\cdot)_{\mathcal{H},-\beta,-t}$. So $F \in \mathcal{H}_t^{\beta}(\mathbb{R}^k)$ can be represented by $F = \sum a_{\mathbf{n}} H_{\mathbf{n}}$. Then for $F = \sum a_{\mathbf{n}} H_{\mathbf{n}}$ and $G = \sum b_{\mathbf{n}} H_{\mathbf{n}}$ in $\mathcal{H}_t^{\beta}(\mathbb{R}^k)$

$$(F,G)_{\mathcal{H},\beta,t} = \sum (\mathbf{n}!)^{1+\beta} e^{2|\mathbf{n}|t} a_{\mathbf{n}} \overline{b_{\mathbf{n}}},$$

$$\|F\|^2_{\mathcal{H},\beta,t} = \sum (\mathbf{n}!)^{1+\beta} e^{2|\mathbf{n}|t} \left|a_{\mathbf{n}}\right|^2.$$

It follows from the inequality, $\|\cdot\|_{\mathcal{H},\beta,s} \leq \|\cdot\|_{\mathcal{H},\beta,t}$, $t \geq s$ that for fixed $\beta \in [0,1)$

$$\mathcal{H}_t^{\beta}(\mathbb{R}^k) \subset \mathcal{H}_s^{\beta}(\mathbb{R}^k)$$

for any $t \geq s$.

$\mathcal{H}_{-t}^{-\beta}(\mathbb{R}^k)$ can be identified as the dual space $(\mathcal{H}_t^{\beta})^*(\mathbb{R}^k)$ of $\mathcal{H}_t^{\beta}(\mathbb{R}^k)$ with the bilinear pairing $\langle\cdot,\cdot\rangle$ given by

$$\langle F, G\rangle \equiv \sum \mathbf{n}! a_{\mathbf{n}} b_{\mathbf{n}}$$

for $F = \sum a_{\mathbf{n}} H_{\mathbf{n}} \in \mathcal{H}_{-t}^{-\beta}(\mathbb{R}^k)$ and $G = \sum b_{\mathbf{n}} H_{\mathbf{n}} \in \mathcal{H}_t^{\beta}(\mathbb{R}^k)$. Let $\mathcal{H}^{\beta}(\mathbb{R}^k)$ be the projective limit of the spaces $\{\mathcal{H}_t^{\beta}(\mathbb{R}^k),\ t \geq 0\}$ and $(\mathcal{H}^{\beta})^*(\mathbb{R}^k)$ be the dual space of $\mathcal{H}^{\beta}(\mathbb{R}^k)$. Then $\mathcal{H}^{\beta}(\mathbb{R}^k) = \cap_{t\geq 0} \mathcal{H}_t^{\beta}(\mathbb{R}^k)$ and $(\mathcal{H}^{\beta})^*(\mathbb{R}^k) = \cup_{t \geq 0} \mathcal{H}_{-t}^{-\beta}(\mathbb{R}^k)$.

Theorem 3.1.2.

For any fixed $\beta \in [0,1)$ the inclusion map i from $\mathcal{H}_t^{\beta}(\mathbb{R}^k)$ into $\mathcal{H}_s^{\beta}(\mathbb{R}^k)$, $t > s \geq 0$, is a Hilbert-Schmidt operator.

Proof.

Since $\|H_{\mathbf{n}}\|^2_{\mathcal{H},\beta,t} = (\mathbf{n}!)^{1+\beta} e^{2|\mathbf{n}|t}$ and $\{H_{\mathbf{n}}\,; \mathbf{n} \geq 0\}$ is a complete orthogonal system in each space $\mathcal{H}_t^{\beta}(\mathbb{R}^k)$, there is an orthonormal basis for $\mathcal{H}_t^{\beta}(\mathbb{R}^k)$

$$\{(\mathbf{n}!)^{-(1+\beta)/2} e^{-|\mathbf{n}|t} H_{\mathbf{n}}\,; \mathbf{n} \geq 0\}.$$

Then

$$\sum \|i(\mathbf{n}!)^{-(1+\beta)/2} e^{-|\mathbf{n}|t} H_{\mathbf{n}}\|^2_{\mathcal{H},\beta,s} = \sum (\mathbf{n}!)^{-1-\beta} e^{-2|\mathbf{n}|t} \|H_{\mathbf{n}}\|^2_{\mathcal{H},\beta,s}$$

$$= \sum e^{2|\mathbf{n}|(s-t)}$$

$$= (1 - e^{2(s-t)})^{-k}. \qquad Q.E.D.$$

It follows from Theorem 3.1.2 that $\mathcal{H}^\beta(\mathbb{R}^k)$ is a nuclear sapce and thus we have the following Gel'fand triple: for fixed $\beta \in [0, 1)$

$$\mathcal{H}^\beta(\mathbb{R}^k) \subset \mathcal{H}_0^0(\mathbb{R}^k) \subset (\mathcal{H}^\beta)^*(\mathbb{R}^k).$$

It is obvious that for any fixed $\beta \in [0, 1)$

$$\mathcal{H}^\beta(\mathbb{R}^k) \subset \mathcal{H}(\mathbb{R}^k),$$

$$\mathcal{H}^*(\mathbb{R}^k) \subset (\mathcal{H}^\beta)^*(\mathbb{R}^k).$$

In particular, $\mathcal{H}^0(\mathbb{R}^k) = \mathcal{H}(\mathbb{R}^k)$, $(\mathcal{H}^0)^*(\mathbb{R}^k) = \mathcal{H}^*(\mathbb{R}^k)$.

Example 3.1.3.

Recall that $\mathcal{H}^\beta(\mathbb{R}^k) \subset \mathcal{H}(\mathbb{R}^k)$. For any given $\beta \in (0, 1)$, choose α such that $\frac{1}{2} < \alpha < \frac{1+\beta}{2}$. Then it can be easily checked by direct computation of norms that

$$F_\alpha = \sum \frac{1}{(\mathbf{n}!)^\alpha} H_{\mathbf{n}} \in \mathcal{H}(\mathbb{R}^k),$$

but $F_\alpha \notin \mathcal{H}^\beta(\mathbb{R}^k)$.

Example 3.1.4.

Recall that $\mathcal{H}^*(\mathbb{R}^k) \subset (\mathcal{H}^\beta)^*(\mathbb{R}^k)$. For any given $\beta \in (0, 1)$, choose α such that $\frac{1-\beta}{2} < \alpha < \frac{1}{2}$. Then it can be also easily checked by direct computation of norms that

$$F_\alpha = \sum \frac{1}{(\mathbf{n}!)^\alpha} H_{\mathbf{n}} \in (\mathcal{H}^\beta)^*(\mathbb{R}^k),$$

but $F_\alpha \notin \mathcal{H}^*(I\!R^k)$.

In the following there are some results about properties of the spaces $\mathcal{H}_t^\beta(I\!R^k)$, $\mathcal{H}_{-t}^{-\beta}(I\!R^k), t \geq 0, \mathcal{H}^\beta(I\!R^k)$ and $(\mathcal{H}^\beta)^*(I\!R^k)$.

Theorem 3.1.5.

Every function F in $\mathcal{H}_t^\beta(I\!R^k), t \geq 0$, is smooth and

$$|F(\mathbf{u})| \leq (1.2)^k \|F\|_{\mathcal{H},\beta,t} \left(\frac{2^\beta - e^{-t} + 2^\beta e^{-t}}{2^\beta - e^{-t}}\right)^k \exp[\mathbf{u}^2/4].$$

Proof.

Suppose that $F \in \mathcal{H}_t^\beta(I\!R^k)$ for fixed $\beta \in [0,1), t \geq 0$, and let F be represented by $F = \sum a_\mathbf{n} H_\mathbf{n}$. Then $e^{-tL}F \in \mathcal{H}_0^\beta(I\!R^k)$, since

$$\begin{aligned} \|e^{-tL}F\|_{\mathcal{H},\beta,0} &= \|e^{-0L}e^{-tL}F\|_{\mathcal{H},\beta} \\ &= \|e^{-tL}F\|_{\mathcal{H},\beta} \\ &= \|F\|_{\mathcal{H},\beta,t} < \infty. \end{aligned}$$

Put $G = e^{-tL}F$. Then we have $F = e^{tL}G$. Note that $\{e^{tL}\}_{t>0}$ is the semigroup of Ornstein-Uhlenbeck process. Therefore F can be expressed in the following form:

$$F(\mathbf{u}) = \int_{R^k} G(\mathbf{v}) Q_t(\mathbf{u}, d\mathbf{v}),$$

where

$$Q_t(\mathbf{u}, d\mathbf{v}) = [\frac{1}{2\pi(1-e^{-2t})}]^{\frac{k}{2}} \exp(-\frac{(\mathbf{v} - e^{-t}\mathbf{u})^2}{2(1-e^{-2t})}) d\mathbf{v}$$

and $d\mathbf{v}$ is the Lebesgue measure on $I\!R^k$. The above integral can be expressed with respect to the standard Gaussian measure μ_k. From the direct computation we get

$$F(\mathbf{u}) = [\frac{1}{(1-e^{-2t})}]^{k/2} \int_{R^k} G(\mathbf{v}) \exp\{-\frac{e^{-2t}\mathbf{u}^2 - 2e^{-t}\mathbf{u}\mathbf{v} + e^{-2t}\mathbf{v}^2}{2(1-e^{-2t})}\} d\mu_k(\mathbf{v}).$$

This implies that F is a smooth function.

Now we show the inequality in the theorem.

$$|F(\mathbf{u})| \leq \sum |a_{\mathbf{n}}||H_{\mathbf{n}}(\mathbf{u})|.$$

By Cramer's estimate for the Hermite functions (see [Erd 53]):

$$|H_n(u)| \leq 1.2(n!)^{1/2}\exp[u^2/4]$$

and Schwarz inequality we have

$$\begin{aligned}
|F(\mathbf{u})| &\leq (1.2)^k(\sum (\mathbf{n}!)^{1/2}|a_{\mathbf{n}}|)\exp[\mathbf{u}^2/4] \\
&= (1.2)^k \left(\sum (\mathbf{n}!)^{(1+\beta)/2} e^{t|\mathbf{n}|}|a_{\mathbf{n}}|(\mathbf{n}!)^{\frac{-\beta}{2}} e^{-t|\mathbf{n}|}\right) \exp[\mathbf{u}^2/4] \\
&\leq (1.2)^k \left(\sum (\mathbf{n}!)^{1+\beta} e^{2t|\mathbf{n}|}|a_{\mathbf{n}}|^2\right)^{1/2} \left(\sum e^{-2t|\mathbf{n}|}(\mathbf{n}!)^{-\beta}\right)^{1/2} \exp[\mathbf{u}^2/4] \\
&= (1.2)^k \|F\|_{\mathcal{H},\beta,t} \left(\sum e^{-2t|\mathbf{n}|}(\mathbf{n}!)^{-\beta}\right)^{1/2} \exp[\mathbf{u}^2/4].
\end{aligned}$$

By using the inequaity $n! \geq 2^{(n-1)}, n \geq 1$, we can obtain the following inequality:

$$\sum_{n=0}^{\infty} \frac{x^n}{(n!)^{\beta}} \leq \frac{2^{\beta} - x + 2^{\beta}x}{2^{\beta} - x}, \quad |x| < 1.$$

By the last inequality, we have

$$|F(\mathbf{u})| \leq (1.2)^k \|F\|_{\mathcal{H},\beta,t} \left(\frac{2^{\beta} - e^{-t} + 2^{\beta}e^{-t}}{2^{\beta} - e^{-t}}\right)^k \exp[\mathbf{u}^2/4]. \qquad Q.E.D.$$

Theorem 3.1.6.

The partial derivatives $\frac{\partial}{\partial u_i}$, $1 \leq i \leq k$, are continuous from $\mathcal{H}^{\beta}(\mathbb{R}^k)$ into $\mathcal{H}^{\beta}(\mathbb{R}^k)$.

Proof.

Suppose that F is in $\mathcal{H}^\beta(\mathbb{R}^k)$ and is represented by $F = \sum a_\mathbf{n} H_\mathbf{n}$. Then

$$\begin{aligned}\frac{\partial}{\partial u_i} F(\mathbf{u}) &= \frac{\partial}{\partial u_i} \sum a_\mathbf{n} H_{n_1}(u_1) \cdots H_{n_k}(u_k) \\ &= \sum a_\mathbf{n} n_i H_{n_1}(u_1) \cdots H_{n_i - 1}(u_i) \cdots H_{n_k}(u_k),\end{aligned}$$

by the fact that

$$\frac{d}{du} H_n(u) = n H_{n-1}(u).$$

Fix $t \geq 0$. Then by letting $(n_i)^2 = (n_i)^{1+\beta}(n_i)^{1-\beta}$ we get

$$\begin{aligned}\|\frac{\partial}{\partial u_i} F\|^2_{\mathcal{H},\beta,t} &= \sum [n_1!]^{1+\beta} \cdots [(n_i - 1)!]^{1+\beta} \cdots [n_k!]^{1+\beta} e^{2(|\mathbf{n}|-1)t} \left|a_\mathbf{n}\right|^2 n_i^2 \\ &= e^{-2t} \sum (\mathbf{n}!)^{1+\beta} e^{2|\mathbf{n}|t} \left|a_\mathbf{n}\right|^2 n_i^{1-\beta}\end{aligned}$$

Choose some $s > t$ and use the fact that $u^{1-\beta} e^{-u} < 1$ and $e^{-u} < 1$ for all $u > 0$ to obtain

$$\begin{aligned}\|\frac{\partial}{\partial u_i} F\|^2_{\mathcal{H},\beta,t} &= e^{-2t} \sum (\mathbf{n}!)^{1+\beta} e^{2|\mathbf{n}|s} \left|a_\mathbf{n}\right|^2 e^{-2|\mathbf{n}|(s-t)} n_i^{1-\beta} \\ &= \frac{e^{-2t}}{[2(s-t)]^{1-\beta}} \sum (\mathbf{n}!)^{1+\beta} e^{2|\mathbf{n}|s} \left|a_\mathbf{n}\right|^2 [2(s-t)]^{1-\beta} n_i^{1-\beta} e^{-2n_i(s-t)} \\ &\quad \times e^{-2n_1(s-t)} \cdots e^{-2n_{i-1}(s-t)} e^{-2n_{i+1}(s-t)} \cdots e^{-2n_k(s-t)} \\ &< \frac{e^{-2t}}{[2(s-t)]^{1-\beta}} \sum (\mathbf{n}!)^{1+\beta} e^{2|\mathbf{n}|s} \left|a_\mathbf{n}\right|^2 \\ &= \frac{e^{-2t}}{[2(s-r)]^{1-\beta}} \|F\|^2_{\mathcal{H},\beta,s}.\end{aligned}$$

Therefore for any $t \geq 0$ we can choose $s > t$ so that

$$\|\frac{\partial}{\partial u_i} F\|_{\mathcal{H},\beta,t} < \frac{e^{-t}}{[2(s-t)]^{(1-\beta)/2}} \|F\|_{\mathcal{H},\beta,s}.$$

This together with the way of construction of the space $\mathcal{H}^\beta(\mathbb{R}^k)$ implies the continuity of the mapping

$$\frac{\partial}{\partial u_i} : \mathcal{H}^\beta(\mathbb{R}^k) \to \mathcal{H}^\beta(\mathbb{R}^k). \qquad Q.E.D.$$

Theorem 3.1.7.

The partial derivatives $\frac{\partial}{\partial u_i}$, $1 \le i \le k$, are continuous from $(\mathcal{H}^\beta)^*(\mathbb{R}^k)$ into $(\mathcal{H}^\beta)^*(\mathbb{R}^k)$.

Proof.

Suppose that F is in $(\mathcal{H}^\beta)^*(\mathbb{R}^k)$ and let F be represented by $F = \sum a_{\mathbf{n}} H_{\mathbf{n}}$. Then by similar calculation in the proof of Theorem 3.1.4, we get

$$\|\frac{\partial}{\partial u_i} F\|^2_{\mathcal{H},-\beta,-t} = e^{2t} \sum (\mathbf{n}!)^{1-\beta} e^{-2|\mathbf{n}|t} \left|a_{\mathbf{n}}\right|^2 n_i^{1+\beta}.$$

We use the fact that $u^{1+\beta}e^{-2u} < 1$ and $e^{-u} < 1$ for all $u > 0$ to obtain the following:

$$\begin{aligned}
\|\frac{\partial}{\partial u_i} F\|^2_{\mathcal{H},-\beta,-t} &= e^{2t} \sum (\mathbf{n}!)^{1-\beta} e^{-2|\mathbf{n}|s} \left|a_{\mathbf{n}}\right|^2 e^{-2|\mathbf{n}|(t-s)} n_i^{1+\beta} \\
&\le \frac{e^{2t}}{(t-s)^{1+\beta}} \sum (n!)^{1-\beta} e^{-2|n|s} \left|a_{\mathbf{n}}\right|^2 e^{-2n_i(t-s)} (t-s)^{1+\beta} n_i^{1+\beta} \\
&\quad \times e^{-2n_1(t-s)} \cdots e^{-2n_{i-1}(t-s)} e^{-2n_{i+1}(t-s)} \cdots e^{-2n_k(t-s)} \\
&< \frac{e^{2t}}{(t-s)^{1+\beta}} \sum (n!)^{1-\beta} e^{-2|n|s} \left|a_{\mathbf{n}}\right|^2 \\
&= \frac{e^{2t}}{(t-s)^{1+\beta}} \|F\|^2_{\mathcal{H},-\beta,-s}.
\end{aligned}$$

So for any $s \ge 0$ we can choose $t > s \ge 0$ so that

$$\|\frac{\partial}{\partial u_i} F\|_{\mathcal{H},-\beta,-t} < \frac{e^t}{(t-s)^{(1+\beta)/2}} \|F\|_{\mathcal{H}-\beta,-s}.$$

This together with the way of construction of the space $(\mathcal{H}^\beta)^*(\mathbb{R}^k)$ implies the continuity of the mapping $\frac{\partial}{\partial u_i} : (\mathcal{H}^\beta)^*(\mathbb{R}^k) \to (\mathcal{H}^\beta)^*(\mathbb{R}^k)$. Q.E.D.

The Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^k)$ is defined by

$$\hat{f}(\mathbf{v}) = \left(\frac{1}{\sqrt{2\pi}}\right)^k \int_{R^k} e^{-i\mathbf{u}\mathbf{v}} f(\mathbf{u}) d\mathbf{u}.$$

It is a well known fact that the Fourier transform can be extended to $\mathcal{S}^*(I\!R^k)$. Recently, the Fourier transform has been extended from $\mathcal{S}^*(I\!R^k)$ to the space $\mathcal{H}^*(I\!R^k)$ in the paper [KK 92]. Recall that $\mathcal{H}^*(I\!R^k) \subset (\mathcal{H}^\beta)^*(I\!R^k)$. We will extend the Fourier transform to the space $(\mathcal{H}^\beta)^*(I\!R^k)$.

For the Fourier transform of fuctions in the space$(\mathcal{H}^\beta)^*(I\!R^k)$, it is enough to consider the Fourier transform of the Hermite polynomials $H_{\mathbf{n}}$ since for any $F \in (\mathcal{H}^\beta)^*(I\!R^k)$, F can be represented by $F = \sum a_{\mathbf{n}} H_{\mathbf{n}}$.

Lemma 3.1.8. [KK 92]

The Fourier transform $\widehat{H_{\mathbf{n}}}$ of $H_{\mathbf{n}}$ is given by

$$\widehat{H_{\mathbf{n}}} = (-i)^{|\mathbf{n}|} \sum_{\mathbf{m}} \frac{(-1)^{|\mathbf{m}|}}{2^{|\mathbf{m}|}\mathbf{m}!} H_{\mathbf{n}+2\mathbf{m}}(\mathbf{v}).$$

Proof We refer to the paper [KK 92].

Lemma 3.1.9.

For any $t > 0$,

$$\|\widehat{H_{\mathbf{n}}}\|_{\mathcal{H},-\beta,-t} \leq (\mathbf{n}!)^{\frac{1-\beta}{2}} e^{\beta k/2} \left(1 - e^{\frac{-2t}{1-\beta}}\right)^{(1-\beta)k/2} \left(e^{\frac{2t}{1-\beta}} - 1\right)^{-(1-\beta)|\mathbf{n}|/2}.$$

Proof

By using the previous lemma and the inequality $2^{2m}(m!)^2 > (2m)!$, we get

$$\begin{aligned}
\|\widehat{H_{\mathbf{n}}}\|^2_{\mathcal{H},-\beta,-t} &= \sum_{\mathbf{m}} \frac{[(\mathbf{n}+2\mathbf{m})!]^{1-\beta}}{2^{2|\mathbf{m}|}(\mathbf{m}!)^2} e^{-2t|\mathbf{n}+2\mathbf{m}|} \\
&\leq \sum_{\mathbf{m}} \frac{[(\mathbf{n}+2\mathbf{m})!]^{1-\beta}}{(2\mathbf{m})!} e^{-2t|\mathbf{n}+2\mathbf{m}|} \\
&\leq \sum_{\mathbf{m}} \frac{[(\mathbf{n}+\mathbf{m})!]^{1-\beta}}{\mathbf{m}!} e^{-2t|\mathbf{n}+\mathbf{m}|} \\
&= e^{-2t|\mathbf{n}|} \sum_{\mathbf{m}} \left[\frac{(\mathbf{n}+\mathbf{m})!}{\mathbf{m}!} e^{\frac{-2t|\mathbf{m}|}{1-\beta}}\right]^{1-\beta} \left[\frac{1}{\mathbf{m}!}\right]^{\beta}.
\end{aligned}$$

Then, by using the Hölder inequality we derive

$$\|\widehat{H_{\mathbf{n}}}\|^2_{\mathcal{H},-\beta,-t} \leq e^{-2t|\mathbf{n}|}\left[\sum_{\mathbf{m}} \frac{(\mathbf{n}+\mathbf{m})!}{\mathbf{m}!} e^{\frac{-2t|\mathbf{m}|}{1-\beta}}\right]^{1-\beta} \left[\sum_{\mathbf{m}} \frac{1}{\mathbf{m}!}\right]^{\beta}.$$

Here we consider the power series $\sum x^n = (1-x)^{-1}, \quad |x| < 1$. By differentiating both sides n times we derive the following equality:

$$\sum_{m=0}^{\infty} \frac{(n+m)!}{m!} x^m = n!(1-x)^{-(n+1)}, \quad |x| < 1.$$

By using the above equality, we get

$$\begin{aligned}
\|\widehat{H_{\mathbf{n}}}\|^2_{\mathcal{H},-\beta,-t} &\leq e^{-2t|\mathbf{n}|}(\mathbf{n}!)^{1-\beta}\left(1-e^{\frac{-2t|\mathbf{m}|}{1-\beta}}\right)^{-(1-\beta)(|\mathbf{n}|+k)} e^{\beta k} \\
&= (\mathbf{n}!)^{1-\beta} e^{\beta k}\left(1-e^{\frac{-2t}{1-\beta}}\right)^{(1-\beta)k}\left(e^{\frac{2t}{1-\beta}}-1\right)^{-(1-\beta)|\mathbf{n}|}.
\end{aligned}$$

Hence we have

$$\|\widehat{H_{\mathbf{n}}}\|_{\mathcal{H},-\beta,-t} \leq (\mathbf{n}!)^{\frac{1-\beta}{2}} e^{\beta k/2}\left(1-e^{\frac{-2t}{1-\beta}}\right)^{(1-\beta)k/2}\left(e^{\frac{2t}{1-\beta}}-1\right)^{-(1-\beta)|\mathbf{n}|/2}.$$

Theorem 3.1.10.

If $F \in (\mathcal{H}^\beta)^*(I\!R^k)$, then its Fourier transfeorm $\hat{F}$ belongs to $(\mathcal{H}^\beta)^*(I\!R^k)$. Moreover, the Fourier transform is continuous from $(\mathcal{H}^\beta)^*(I\!R^k)$ into $(\mathcal{H}^\beta)^*(I\!R^k)$.

Proof

Let $F \in (\mathcal{H}^\beta)^*(I\!R^k)$ and be represented by $F = \sum_{\mathbf{n}} a_{\mathbf{n}} H_{\mathbf{n}}$. Then by Lemma

3.1.8, we have

$$\begin{aligned}
\|\hat{F}\|_{\mathcal{H},-\beta,-t} &\leq \sum_{\mathbf{n}} |a_{\mathbf{n}}| \|\widehat{H_{\mathbf{n}}}\|_{\mathcal{H},-\beta,-t} \\
&\leq \sum_{\mathbf{n}} |a_{\mathbf{n}}| (\mathbf{n}!)^{\frac{1-\beta}{2}} e^{\beta k/2} \left(1 - e^{\frac{-2t}{1-\beta}}\right)^{(1-\beta)k/2} \left(e^{\frac{2t}{1-\beta}} - 1\right)^{-(1-\beta)|\mathbf{n}|/2} \\
&\leq e^{\beta k/2} \left(1 - e^{\frac{-2t}{1-\beta}}\right)^{(1-\beta)k/2} \left(\sum \left(e^{\frac{2t}{1-\beta}} - 1\right)^{-(1-\beta)|\mathbf{n}|} e^{2s|\mathbf{n}|}\right)^{1/2} \\
&\quad \times \left(\sum (\mathbf{n}!)^{1-\beta} e^{-2s|\mathbf{n}|} |a_{\mathbf{n}}|^2\right)^{1/2} \\
&= e^{\beta k/2} \left(1 - e^{\frac{-2t}{1-\beta}}\right)^{(1-\beta)k/2} \\
&\quad \times \left(\sum \left(e^{\frac{2t}{1-\beta}} - 1\right)^{-(1-\beta)|\mathbf{n}|} e^{2s|\mathbf{n}|}\right)^{1/2} \|F\|_{\mathcal{H},-\beta,-s}.
\end{aligned}$$

Thus, for any given $s > 0$, we can choose $t > 0$ large enough such that

$$\left(e^{\frac{2t}{1-\beta}} - 1\right)^{1-\beta} > e^{2s}.$$

Then we have

$$\begin{aligned}
\|\hat{F}\|_{\mathcal{H},-\beta,-t} &\leq e^{\beta k/2} \left(1 - e^{\frac{-2t}{1-\beta}}\right)^{(1-\beta)k/2} \left(1 - \left(e^{\frac{2t}{1-\beta}} - 1\right)^{1-\beta} e^{2s}\right)^{-k/2} \|F\|_{\mathcal{H},-\beta,-s} \\
&= e^{\beta k/2} \left(\left(1 - e^{\frac{-2t}{1-\beta}}\right)^{1-\beta} - e^{-2(t-s)}\right)^{-k/2} \|F\|_{\mathcal{H},-\beta,-s}.
\end{aligned}$$

Therefore, $\widehat{F} \in (\mathcal{H}^{\beta})^*(\mathbb{R}^k)$ for any $F \in (\mathcal{H}^{\beta})^*(\mathbb{R}^k)$ and the Fourier transform is continuous form $(\mathcal{H}^{\beta})^*(\mathbb{R}^k)$ into itself. Q.E.D.

In the following the elements in $\mathcal{H}^{\beta}(\mathbb{R}^k)$ and in $(\mathcal{H}^{\beta})^*(\mathbb{R}^k)$ are characterized in the same way as in [KK 92]. Thus for these characterizations it is necessary to introduce another Gel'fand triple

$$\mathcal{F}^{\beta}(\mathbb{R}^k) \subset \mathcal{F}_0^0(\mathbb{R}^k) \subset (\mathcal{F}^{\beta})^*(\mathbb{R}^k)$$

and to extend the σ-transform to $(\mathcal{H}^{\beta})^*(\mathbb{R}^k)$.

For $F(\mathbf{u}) = \sum_{\mathbf{n}} a_{\mathbf{n}}\mathbf{u}^{\mathbf{n}}, G(\mathbf{u}) = \sum_{\mathbf{n}} b_{\mathbf{n}}\mathbf{u}^{\mathbf{n}} \in \mathcal{P}(I\!R^k)$ define another inner product $(\cdot,\cdot)_{\mathcal{F},\beta}$ on $\mathcal{P}(I\!R^k)$ by

$$(F,G)_{\mathcal{F},\beta} \equiv \sum_{\mathbf{n}} (\mathbf{n}!)^{1+\beta} a_{\mathbf{n}} \overline{b_{\mathbf{n}}}.$$

Denote $\|\cdot\|_{\mathcal{F},\beta}$ to be the norm corresponding to the inner product $(\cdot,\cdot)_{\mathcal{F},\beta}$, then

$$\|F\|^2_{\mathcal{F},\beta} \equiv \sum_{\mathbf{n}} (\mathbf{n}!)^{1+\beta} |a_{\mathbf{n}}|^2.$$

For each $t \geq 0$, define another inner product $(\cdot,\cdot)_{\mathcal{F},\beta,t}$ on $\mathcal{P}(I\!R^k)$ by

$$(F,G)_{\mathcal{F},\beta,t} \equiv \sum (\mathbf{n}!)^{1+\beta} e^{2|\mathbf{n}|t} a_{\mathbf{n}} \overline{b_{\mathbf{n}}}.$$

Denote $\|\cdot\|_{\mathcal{F},\beta,t}$ to be the norm corresponding to the inner product $(\cdot,\cdot)_{\mathcal{F},\beta,t}$. Then

$$\|F\|^2_{\mathcal{F},\beta,t} \equiv \sum (\mathbf{n}!)^{1+\beta} e^{2|\mathbf{n}|t} |a_{\mathbf{n}}|^2.$$

For each $t \geq 0$, define $\mathcal{F}^{\beta}_t(I\!R^k)$ to be the completion of $\mathcal{P}(I\!R^k)$ with respect to the norm $\|\cdot\|_{\mathcal{F},\beta,t}$ and $\mathcal{F}^{-\beta}_{-t}(I\!R^k)$ to be the completion of $\mathcal{P}(I\!R^k)$ with respect to the norm $\|\cdot\|_{\mathcal{F},-\beta,-t}$ corresponding to the inner product $(\cdot,\cdot)_{\mathcal{F},-\beta,-t}$ given by

$$(F,G)_{\mathcal{F},-\beta,-t} \equiv \sum (\mathbf{n}!)^{1-\beta} e^{-2|\mathbf{n}|t} a_{\mathbf{n}} \overline{b_{\mathbf{n}}}.$$

It is clear that the collection $\{\mathbf{u}^{\mathbf{n}}; \mathbf{n} \geq 0\}$ is a complete orthogonal system in each $\mathcal{F}^{\beta}_t(I\!R^k)$ and $\mathcal{F}^{-\beta}_{-t}(I\!R^k)$. Also it is easy to see that

$$\mathcal{F}^{\beta}_t(I\!R^k) \subset \mathcal{F}^{\beta}_s(I\!R^k)$$

for any $t \geq s \geq 0$. $\mathcal{F}^{-\beta}_{-t}(I\!R^k)$ can be identified as the dual space $(\mathcal{F}^{\beta}_t)^*(I\!R^k)$ of $\mathcal{F}^{\beta}_t(I\!R^k)$ with respect to the bilinear pairing $\langle\cdot,\cdot\rangle$ given by

$$\langle F,G\rangle \equiv \sum_{\mathbf{n}} \mathbf{n}! a_{\mathbf{n}} b_{\mathbf{n}}$$

for $F = \sum a_{\mathbf{n}}\mathbf{u}^{\mathbf{n}} \in \mathcal{F}_{-t}^{-\beta}(\mathbb{R}^k)$, $G = \sum b_{\mathbf{n}}\mathbf{u}^{\mathbf{n}} \in \mathcal{F}_t^{\beta}(\mathbb{R}^k)$. Define $\mathcal{F}^{\beta}(\mathbb{R}^k)$ to be the projective limit of the spaces $\{\mathcal{F}_t^{\beta}(\mathbb{R}^k); t \geq 0\}$ and $(\mathcal{F}^{\beta})^*(\mathbb{R}^k)$ the dual space of $\mathcal{F}^{\beta}(\mathbb{R}^k)$. Then $\mathcal{F}^{\beta}(\mathbb{R}^k) = \cap_{t\geq 0}\mathcal{F}_t^{\beta}(\mathbb{R}^k)$ and $(\mathcal{F}^{\beta})^*(\mathbb{R}^k) = \cup_{t\geq 0}\mathcal{F}_{-t}^{-\beta}(\mathbb{R}^k)$. It can be easily checked that the inclusion map i from $\mathcal{F}_t^{\beta}(\mathbb{R}^k)$ into $\mathcal{F}_s^{\beta}(\mathbb{R}^k)$, $t > s$, is a Hilbert-Schmidt operator by using the similar way to the proof of Fact 3.1.2. Therefore $\mathcal{F}^{\beta}(\mathbb{R}^k)$ is a nuclear sapce and we have the following Gel'fand triple: for fixed $\beta \in [0, 1)$

$$\mathcal{F}^{\beta}(\mathbb{R}^k) \subset \mathcal{F}_0^0(\mathbb{R}^k) \subset (\mathcal{F}^{\beta})^*(\mathbb{R}^k).$$

It is obvious that for any fixed $\beta \in [0, 1)$

$$\mathcal{F}^{\beta}(\mathbb{R}^k) \subset \mathcal{F}(\mathbb{R}^k),$$

$$\mathcal{F}^*(\mathbb{R}^k) \subset (\mathcal{F}^{\beta})^*(\mathbb{R}^k).$$

In particular, $\mathcal{F}^0(\mathbb{R}^k) = \mathcal{F}(\mathbb{R}^k)$, $(\mathcal{F}^0)^*(\mathbb{R}^k) = \mathcal{F}^*(\mathbb{R}^k)$.

Now , recall the notation $e_{\mathbf{v}}(\mathbf{u}) = \exp(\mathbf{uv} - \mathbf{v}^2/2)$ from §2.1. In order to extend the σ-transform to the space $(\mathcal{H}^{\beta})^*(\mathbb{R}^k)$ it is necessary to check the following

Fact 3.1.11.

For all $\mathbf{v} \in \mathbb{R}^k, e_{\mathbf{v}} \in \mathcal{H}^{\beta}(\mathbb{R}^k)$.

Proof. For any $t \geq 0$, we have

$$\begin{aligned}\|e_{\mathbf{v}}\|^2_{\mathcal{H},\beta,t} &= \sum (\mathbf{n}!)^{1+\beta} e^{2|\mathbf{n}|t} \frac{\mathbf{v}^{2\mathbf{n}}}{(\mathbf{n}!)^2} \\ &= \sum e^{2|\mathbf{n}|t} \frac{\mathbf{v}^{2\mathbf{n}}}{(\mathbf{n}!)^{1-\beta}} < \infty\end{aligned}$$

by the ratio test. By the construction of the space $\mathcal{H}^{\beta}(\mathbb{R}^k)$, $e_{\mathbf{v}} \in \mathcal{H}^{\beta}(\mathbb{R}^k)$. Q.E.D.

By Fact 3.1.3 we can extend the σ-transform from the space $\mathcal{H}^*(\mathbb{R}^k)$ to the space $(\mathcal{H}^\beta)^*(\mathbb{R}^k)$ as follows.

Definition 3.1.12.

The σ-transform of $F \in (\mathcal{H}^\beta)^*(\mathbb{R}^k)$ is defined by

$$(\sigma F)(\mathbf{v}) \equiv \langle F, e_{\mathbf{v}} \rangle, \qquad \mathbf{v} \in \mathbb{R}^k.$$

By applying the same arguments for Theorem 2.1.4 and Corollary 2.1.5 to the spaces $\mathcal{H}_t^\beta(\mathbb{R}^k)$, $\mathcal{H}_{-t}^{-\beta}(\mathbb{R}^k)$, $t \geq 0$, the corresponding results can be obtained as follows.

Theorem 3.1.13.

The σ-transform is an isometry from $\mathcal{H}_t^\beta(\mathbb{R}^k)$ onto $\mathcal{F}_t^\beta(\mathbb{R}^k)$ and also from $\mathcal{H}_{-t}^{-\beta}(\mathbb{R}^k)$ onto $\mathcal{F}_{-t}^{-\beta}(\mathbb{R}^k)$ for each $t \geq 0$.

Proof.

Suppose that $F = \sum a_{\mathbf{n}} H_{\mathbf{n}} \in \mathcal{H}_t^\beta(\mathbb{R}^k)$ and $G = \sum b_{\mathbf{n}} H_{\mathbf{n}} \in (\mathcal{H}_t^\beta)^*(\mathbb{R}^k)$. Then recall from §2.1 that $(\sigma F)(\mathbf{v}) = \sum a_{\mathbf{n}} \mathbf{v}^{\mathbf{n}}$ and $(\sigma G)(\mathbf{v}) = \sum b_{\mathbf{n}} \mathbf{v}^{\mathbf{n}}$. Then we immediately see that

$$\|F\|^2_{\mathcal{H},\beta,t} = \sum_{\mathbf{n}} (\mathbf{n}!)^{1+\beta} e^{2|\mathbf{n}|t} \, |a_{\mathbf{n}}|^2 = \|\sigma F\|^2_{\mathcal{F},\beta,t}$$

$$\|G\|^2_{\mathcal{H},-\beta,-t} = \sum_{\mathbf{n}} (\mathbf{n}!)^{1-\beta} e^{2|\mathbf{n}|t} \, |b_{\mathbf{n}}|^2 = \|\sigma G\|^2_{\mathcal{F},-\beta,-t}.$$

It is clear that σ is onto for each $t \geq 0$ from the representations of F, σF, G and σG. Q.E.D.

Furthermore, it follows immediately from Theorem 3.1.11 that the σ-transform is continuous from $\mathcal{H}_t^\beta(\mathbb{R}^k)$ onto $\mathcal{F}_t^\beta(\mathbb{R}^k)$ and also from $\mathcal{H}_{-t}^{-\beta}(\mathbb{R}^k)$ onto $\mathcal{F}_{-t}^{-\beta}(\mathbb{R}^k)$ for each $t \geq 0$. Thus we have the following

Corollary 3.1.14.

The σ-transform is homeomorphic from $\mathcal{H}^{\beta}(\mathbb{R}^k)$ onto $\mathcal{F}^{\beta}(\mathbb{R}^k)$ and also from $(\mathcal{H}^{\beta})^*(\mathbb{R}^k)$ onto $(\mathcal{F}^{\beta})^*(\mathbb{R}^k)$.

In the following the characterization theorems for $\mathcal{F}^{\beta}(\mathbb{R}^k)$ and $(\mathcal{F}^{\beta})^*(\mathbb{R}^k)$ are introduced. These theorems are generalizations of Theorem 2.1.6 and Theorem 2.1.7 (with $\beta = 0$). These theorems can be regarded as the finite dimensional version of Theorem 1.2.3 and Theorem 1.2.4.

Theorem 3.1.15.

A function F belongs to $\mathcal{F}^{\beta}(\mathbb{R}^k)$ if and only if $F(\mathbf{u})$, $\mathbf{u} \in \mathbb{R}^k$, has an entire extension $F(\mathbf{z})$, $\mathbf{z} \in \mathbb{C}^k$, and satisfies the following condition: for any $a > 0$ there exists a constant $A > 0$ such that

$$|F(\mathbf{z})| \leq A e^{a|\mathbf{z}|^{\frac{2}{1+\beta}}}, \qquad \mathbf{z} \in \mathbb{C}^k.$$

Proof.

First suppose that $F \in \mathcal{F}^{\beta}(\mathbb{R}^k)$ and F is represented by $F(\mathbf{u}) = \sum a_{\mathbf{n}} \mathbf{u}^{\mathbf{n}}$. In order to show that $F(\mathbf{z})$ is an entire funcion on $\mathbb{C}^k$ we check the convergence of the power series $F(\mathbf{z}) = \sum a_{\mathbf{n}} \mathbf{z}^{\mathbf{n}}$, $\mathbf{z} \in \mathbb{C}^k$. Since $F(\mathbf{u}) \in \mathcal{F}^{\beta}(\mathbb{R}^k)$, for any $t \geq 0$ we have

$$\|F\|^2_{\mathcal{F},\beta,t} = \sum (\mathbf{n}!)^{1+\beta} e^{2|\mathbf{n}|t} |a_{\mathbf{n}}|^2 < \infty.$$

Thus for each $\mathbf{n} \geq 0$

$$|a_{\mathbf{n}}|^2 \leq (\mathbf{n}!)^{-1-\beta} e^{-2|\mathbf{n}|t} \|F\|^2_{\mathcal{F},\beta,t},$$

$$|a_{\mathbf{n}}| \leq (\mathbf{n}!)^{\frac{-1-\beta}{2}} e^{-|\mathbf{n}|t} \|F\|_{\mathcal{F},\beta,t}.$$

Thus we get the following:

$$
\begin{aligned}
|F(\mathbf{z})| &\le \sum (\mathbf{n}!)^{\frac{-1-\beta}{2}} e^{-|\mathbf{n}|t} \|F\|_{\mathcal{F},\beta,t} |z_1|^{n_1} \cdots |z_k|^{n_k} \\
&= \|F\|_{\mathcal{F},\beta,t} \prod_{i=1}^{k} \sum_{n_i} (n_i!)^{\frac{-1-\beta}{2}} e^{n_i r} |z_i|^{n_i}.
\end{aligned}
$$

Then for any $t \ge 0$ we choose any $0 \le s < r$ and then rewrite the above expression as

$$
|F(\mathbf{z})| \le \|F\|_{\mathcal{F},\beta,t} \prod_{i=1}^{k} \sum_{n_i} (n_i!)^{\frac{-1-\beta}{2}} e^{-n_i s} e^{(s-r)n_i} |z_i|^{n_i}.
$$

Then by the Schwarz inequality we obtain

$$
|F(\mathbf{z})| \le \|F\|_{\mathcal{F},\beta,t} \prod_{i=1}^{k} \left(\sum_{n_i} e^{-2n_i s} \right)^{1/2} \prod_{i=1}^{k} \left(\sum \frac{(e^{2(s-t)} |z_i|^2)^{n_i}}{(n_i!)^{1+\beta}} \right)^{1/2}.
$$

The following inequality

$$
\left| \sum \frac{u^n}{(n!)^{1+\beta}} \right| \le C_\beta \exp[2u^{1/(1+\beta)}], \qquad u > 0
$$

with C_β a constant depending on β has been used in the paper [KS 92]. Use this inequality to get the following:

$$
\begin{aligned}
|F(\mathbf{z})| &\le \|F\|_{\mathcal{F},\beta,t} (1 - e^{-2s})^{-k/2} \prod_{i=1}^{k} C_\beta \exp\left[\left(\{\exp(2s - 2t)\} |z_i|^2 \right)^{1/(1+\beta)} \right] \\
&= \|F\|_{\mathcal{F},\beta,t} (1 - e^{-2s})^{-k/2} C_\beta^k \exp\left[\exp(\frac{2s - 2t}{1+\beta}) |\mathbf{z}|^{2/(1+\beta)} \right].
\end{aligned}
$$

Therefore we conclude not only that the power series $F(\mathbf{z}) = \sum_n a_{\mathbf{n}} \mathbf{z}^{\mathbf{n}}$, $\mathbf{z} \in \mathbb{C}^k$, converges for all $\mathbf{z} \in \mathbb{C}^k$, and thus $F(\mathbf{z})$ is an entire function but also that $F(\mathbf{z})$ satisfies the condition in this theorem.

On the other hand, suppose that $F(\mathbf{z})$ is an entire function on $\mathbb{C}^k$ such that for any $a > 0$, there exists $A > 0$ such that

$$
|F(\mathbf{z})| \le A e^{a|\mathbf{z}|^{2/(1+\beta)}}, \qquad \mathbf{z} \in \mathbb{C}^k.
$$

Then we can represent $F(\mathbf{z})$ as a power series $F(\mathbf{z}) = \sum a_{\mathbf{n}}\mathbf{z}^{\mathbf{n}}$ on $\mathbb{C}^k$ since $F(\mathbf{z})$ is entire on $\mathbb{C}^k$. To show $F(\mathbf{u}) \in \mathcal{F}^{\beta}(\mathbb{R}^k)$, we need to show that $\|F\|_{\mathcal{F},\beta,t} < \infty$ for all $t \geq 0$. Thus consider

$$|F|^2_{\mathcal{F},\beta,t} = \sum (\mathbf{n}!)^{1+\beta} e^{2|\mathbf{n}|t} |a_{\mathbf{n}}|^2 .$$

By the Cauchy formula and the inequality in the assumption we have the followings:

$$\begin{aligned}
|a_{\mathbf{n}}| &= \left| \left(\frac{1}{2\pi i}\right)^k \int_{|z_i|=r_1} \cdots \int_{|z_k|=r_k} \frac{F(\mathbf{z})}{\mathbf{z}^{\mathbf{n}+1}} dz_1 \cdots dz_k \right| \\
&\leq \left(\frac{1}{2\pi}\right)^k \int_{|z_i|=r_1} \cdots \int_{|z_k|=r_k} \frac{|F(\mathbf{z})|}{|\mathbf{z}|^{\mathbf{n}+1}} |dz_1| \cdots |dz_k| \\
&\leq A \left(\frac{1}{2\pi}\right)^k \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{\exp[a\mathbf{r}^{2/(1+\beta)}]}{\mathbf{r}^{\mathbf{n}+1}} \mathbf{r} d\theta \cdots d\theta \\
&= A \prod_{i=1}^k \frac{1}{r_i^{n_i}} \exp(a r_i^{2/(1+\beta)}).
\end{aligned}$$

Then by minimizing the last expression we can get the best estimation for $|a_{\mathbf{n}}|$. From the direct computation, we can see that the function $r^{-n}\exp[ar^{2/(1+\beta)}]$, $r > 0$, attains the minimum value at $r = [\frac{n(1+\beta)}{2a}]^{(1+\beta)/2}$. Therefore we choose $r_i = [\frac{n_i(1+\beta)}{2a}]^{(1+\beta)/2}$ if $n_i \geq 1$. For the case $n_i = 0$ we choose any $r_i > 0$ and let r_i go to zero. Then, with setting $\tau = \frac{1+\beta}{2}$ for simplicity and the convention $0^0 = 1$, we have

$$\begin{aligned}
|a_{\mathbf{n}}| &\leq A \prod_{i=1}^k \frac{a^{n_i\tau} e^{n_i\tau}}{(n_i\tau)^{n_i\tau}} \\
&= A \prod_{i=1}^k \left[\left(\frac{e}{n_i}\right)^{n_i}\right]^\tau \frac{a^{n_i\tau}}{(\tau^\tau)^{n_i}}.
\end{aligned}$$

By the Stirling's formula $n! \sim \sqrt{2\pi n}(n/e)^n$, there exists a constant C such that

$$n! \leq C\sqrt{2\pi n} e^{-n} n^n.$$

Thus the following inequality holds.

$$|a_{\mathbf{n}}| \le AC^k \prod_{i=1}^{k} \left(\frac{\sqrt{2\pi n_i}}{n_i!}\right)^{\tau} a^{n_i\tau} \frac{1}{(\tau^{\tau})^{n_i}}.$$

By using the fact that $\tau \in [1/2, 1)$ and $\tau^{\tau} \in [1/\sqrt{2}, 1)$ we obtain

$$\begin{aligned} |a_{\mathbf{n}}| &\le AC^k \prod_{i=1}^{k} \sqrt{2\pi n_i} a^{n_i\tau} \sqrt{2}^{n_i} \frac{1}{(n_i!)^{\tau}} \\ &= AC^k (2\pi)^{k/2} \mathbf{n}^{1/2} 2^{\mathbf{n}/2} a^{|\mathbf{n}|\tau} (\mathbf{n}!)^{-\tau}. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} |a_{\mathbf{n}}|^2 &\le A^2 \mathbf{C}^2 (2\pi)^k \mathbf{n} 2^{\mathbf{n}} a^{2|\mathbf{n}|\tau} (\mathbf{n}!)^{-2\tau} \\ &= A^2 C^{2k} (2\pi)^k \mathbf{n} 2^{\mathbf{n}} a^{|\mathbf{n}|(1+\beta)} (\mathbf{n}!)^{-1-\beta}. \end{aligned}$$

From this estimation,

$$\begin{aligned} \|F\|^2_{\mathcal{F},\beta,t} &= \sum (\mathbf{n}!)^{1+\beta} e^{2|\mathbf{n}|t} |a_{\mathbf{n}}|^2 \\ &\le A^2 C^{2k} (2\pi)^k \sum (2a^{1+\beta})^{|\mathbf{n}|} e^{2|\mathbf{n}|t} \mathbf{n} \\ &= A^2 C^{2k} (2\pi)^k \sum \mathbf{n} (2a^{1+\beta} e^{2t})^{|\mathbf{n}|}. \end{aligned}$$

For the last summation use the following fact that for $|u| < 1$, $\sum_{n=1}^{\infty} nu^n = u/(1-u)^2$. So for any given $t \ge 0$, choose $a > 0$ such that $2a^{1+\beta}e^{2t} < 1$. Then

$$\|F\|_{\mathcal{F},\beta,t} \le AC^k (2\pi)^{k/2} (2a^{1+\beta}e^{2t})^{k/2} (1 - 2a^{1+\beta}e^{2t})^{-k}.$$

Hence F is in $\mathcal{F}_t^{\beta}(\mathbb{R}^k)$ for any $t \ge 0$. Therefore, the definition of the space $\mathcal{F}^{\beta}(\mathbb{R}^k) = \cap_{r\ge 0} \mathcal{F}_t^{\beta}(\mathbb{R}^k)$ yields that F is in $\mathcal{F}^{\beta}(\mathbb{R}^k)$.

Theorem 3.1.16.

A function F is in $(\mathcal{F}^{\beta})^*(\mathbb{R}^k)$ if and only if $F(\mathbf{u})$, $\mathbf{u} \in \mathbb{R}^k$, has an entire extension $F(\mathbf{z})$, $\mathbf{z} \in \mathbb{C}^k$, and there exist constants $b > 0$ and $A > 0$ such that

$$|F(\mathbf{z})| \le A \exp[b\,|\mathbf{z}|^{2/(1-\beta)}],\ \mathbf{z} \in \mathbb{C}^k.$$

Proof.

First suppose that F is in $(\mathcal{F}^\beta)^*(\mathbb{R}^k)$ and let $F = \sum a_{\mathbf{n}} \mathbf{u}^{\mathbf{n}}$. Then for some $t \geq 0$

$$\|F\|^2_{\mathcal{F},-\beta,-t} = \sum (\mathbf{n}!)^{1-\beta} e^{-2|\mathbf{n}|r} |a_{\mathbf{n}}|^2 < \infty.$$

By the similar calculation in the proof of Theorem 3.1.7, we get the following for the power series $F(\mathbf{z}) = \sum a_{\mathbf{n}} \mathbf{z}^{\mathbf{n}}$, $\mathbf{z} \in \mathbb{C}$: For any $s > t$

$$|F(\mathbf{z})| \leq \|F\|_{\mathcal{F},-\beta,-t} (1 - e^{-2s})^{-k/2} \prod_{i=1}^{k} \left(\sum_{n_i} \frac{(e^{2(t-s)} |z_i|^2)^{n_i}}{(n_i!)^{1-\beta}} \right)^{1/2}.$$

By using the Hölder inequality we can derive the following inequality:

$$\sum \frac{u^n}{(n!)^{1-\beta}} \leq \left(1 - d^{\frac{-1}{\beta}}\right)^{-\beta} \exp\left[(1-\beta) d^{\frac{1}{1-\beta}} u^{\frac{1}{1-\beta}}\right]$$

for any $d > 1$. By the above inequality we obtain the following:

$$|F(\mathbf{z})| \leq \|F\|_{\mathcal{F},-\beta,-t} (1 - e^{-2s})^{-k/2} \left(1 - d^{\frac{-1}{\beta}}\right)^{-\beta k/2}$$
$$\times \exp\left[\left(\frac{1-\beta}{2}\right) d^{\frac{1}{1-\beta}} \exp\left(\frac{2t - 2s}{1-\beta}\right) |\mathbf{z}|^{\frac{2}{1-\beta}} \right].$$

Hence, the power series $F(\mathbf{z}) = \sum a_{\mathbf{n}} \mathbf{z}^{\mathbf{n}}$, $\mathbf{z} \in \mathbb{C}$, converges for all $\mathbf{z} \in \mathbb{C}$, and thus $F(\mathbf{z})$ is an entire function. Moreover $F(\mathbf{z})$ satisfies the condition in the theorem.

Conversely, assume that $F(\mathbf{z})$ is an entire function on $\mathbb{C}$ such that there exist non-negative contants b and A satisfying

$$|F(\mathbf{z})| \leq A \exp[b\, |\mathbf{z}|^{2/(1-\beta)}], \qquad \mathbf{z} \in \mathbb{C}.$$

The similar calculation by replacing $-\beta$ with β in the proof of Theorem 3.1.13 yields

$$|a_{\mathbf{n}}|^2 \leq A^2 C^{2k} b^{|\mathbf{n}|(1-\beta)} 2^{\mathbf{n}} (2\pi)^k \mathbf{n} (\mathbf{n}!)^{-1+\beta},$$

where $n! \leq C\sqrt{2\pi n}e^{-n}n^n$. Therefore, we choose $t > 0$ such that $0 < 2b^{1-\beta}e^{-2t} < 1$ to obtain

$$\|F\|_{\mathcal{F},-\beta,-t} \leq A(2\pi)^{k/2}C^k(1-2b^{1-\beta}e^{-2r})^{-k} < \infty.$$

Hence we conclude that F is in $\mathcal{F}_{-t}^{-\beta}(\mathbb{R}^k)$ for some $t \geq 0$ and thus that F is in $(\mathcal{F}^\beta)^*(\mathbb{R}^k)$ by the construction of the space $(\mathcal{F}^\beta)^*(\mathbb{R}^k)$. Q.E.D.

Theorem 3.1.17.

$\mathcal{H}^\beta(\mathbb{R}^k) = \mathcal{F}^\beta(\mathbb{R}^k)$ as sets.

Proof.

We first show that $\mathcal{H}^\beta(\mathbb{R}^k) \subset \mathcal{F}^\beta(\mathbb{R}^k)$. Suppose that $F \in \mathcal{H}^\beta(\mathbb{R}^k)$ is represented by $F(\mathbf{u}) = \sum_{\mathbf{n}} a_{\mathbf{n}}H_{\mathbf{n}}(\mathbf{u})$.

$$\begin{aligned} F(\mathbf{u}) &= \sum_{\mathbf{n}} a_{\mathbf{n}} \sum_{2\mathbf{j}\leq\mathbf{n}} \frac{(-1)^{\mathbf{j}}\mathbf{n}!2^{-\mathbf{j}}}{\mathbf{j}!(\mathbf{n}-2\mathbf{j})!}\mathbf{u}^{\mathbf{n}-2\mathbf{j}} \\ &= \sum_{\mathbf{j}}\sum_{\mathbf{m}} a_{\mathbf{m}+2\mathbf{j}} \frac{(-1)^{\mathbf{j}}(\mathbf{m}+2\mathbf{j})!2^{-\mathbf{j}}}{\mathbf{j}!\mathbf{m}!}\mathbf{u}^{\mathbf{m}} \\ &= \sum_{\mathbf{m}}\left(\sum_{\mathbf{j}} a_{\mathbf{m}+2\mathbf{j}} \frac{(-1)^{\mathbf{j}}(\mathbf{m}+2\mathbf{j})!2^{-\mathbf{j}}}{\mathbf{j}!\mathbf{m}!}\right)\mathbf{u}^{\mathbf{m}}. \end{aligned}$$

By the Schwarz inequality, for any $t \geq 0$ we have

$$\begin{aligned} \|F\|^2_{\mathcal{F},\beta,t} &= \sum_{\mathbf{m}} (\mathbf{m}!)^{1+\beta}e^{2t|\mathbf{m}|}\left|\sum_{\mathbf{j}} a_{\mathbf{m}+2\mathbf{j}}(-1)^{\mathbf{j}}(\mathbf{m}+2\mathbf{j})!\frac{2^{-\mathbf{j}}}{\mathbf{j}!}\mathbf{m}!\right|^2 \\ &= \sum_{\mathbf{m}} \frac{1}{(\mathbf{m}!)^{1-\beta}}e^{2t|\mathbf{m}|}\left|\sum_{\mathbf{j}} a_{\mathbf{m}+2\mathbf{j}}\frac{(-1)^{\mathbf{j}}(\mathbf{m}+2\mathbf{j})!2^{-\mathbf{j}}}{\mathbf{j}!}\right|^2. \end{aligned}$$

Let us denote $C_{\mathbf{m}}$ the summation on $\mathbf{j}$ above. Then

$$C_{\mathbf{m}} = \sum_{\mathbf{j}} a_{\mathbf{m}+2\mathbf{j}}e^{|\mathbf{m}+2\mathbf{j}|_s}[(\mathbf{m}+2\mathbf{j})!]^{(1+\beta)/2}(-1)^{\mathbf{j}}\frac{2^{-\mathbf{j}}}{\mathbf{j}!}[(\mathbf{m}+2\mathbf{j})!]^{(1-\beta)/2}e^{-|\mathbf{m}+2\mathbf{j}|_s}.$$

Use the Schwarz inequality and the fact that $(2j)! \leq 2^{2j}(j!)^2$. Then we have

$$|C_{\mathbf{m}}|^2 \leq e^{-2\mathbf{m}s}\sum_{\mathbf{j}}|a_{\mathbf{m}+2\mathbf{j}}|^2\, e^{2|\mathbf{m}+2\mathbf{j}|s}[(\mathbf{m}+2\mathbf{j})!]^{1+\beta}\sum_{\mathbf{j}}\frac{[(\mathbf{m}+2\mathbf{j})!]^{1-\beta}}{(2\mathbf{j})!}e^{-4|\mathbf{j}|s}$$
$$\leq e^{-2\mathbf{m}s}\|F\|^2_{\mathcal{H}_{\beta,r}}\sum_{\mathbf{j}}\frac{[(\mathbf{m}+2\mathbf{j})!]^{1-\beta}}{(2\mathbf{j})!}e^{-2|\mathbf{j}|s}.$$

So for any $r \geq 0$, choose $s > 0$ such that $e^{-2s}+e^{2(r-s)} < 1$. Then

$$\|F\|^2_{\mathcal{F},\beta,r} \leq \|F\|^2_{\mathcal{H},\beta,r}\sum_{\mathbf{m}}\frac{e^{2|\mathbf{m}|(r-s)}}{(\mathbf{m}!)^{1-\beta}}\sum_{\mathbf{j}}\frac{[(\mathbf{m}+\mathbf{j})!]^{1-\beta}}{\mathbf{j}!}e^{-2|\mathbf{j}|s}$$
$$= \|F\|^2_{\mathcal{H},\beta,r}\sum_{\mathbf{m}}\sum_{\mathbf{j}}\left[\frac{(\mathbf{m}+\mathbf{j})!}{\mathbf{m}!}\right]^{1-\beta}\frac{e^{2|\mathbf{m}|(r-s)}e^{-2|\mathbf{j}|s}}{\mathbf{j}!}.$$

Since

$$\frac{(\mathbf{m}+\mathbf{j})!}{\mathbf{m}!} > \left(\frac{(\mathbf{m}+\mathbf{j})!}{\mathbf{m}!}\right)^{1-\beta},$$

we have

$$\|F\|^2_{\mathcal{F},\beta,r} \leq \|F\|^2_{\mathcal{H},\beta,r}\sum\sum\frac{(\mathbf{m}+\mathbf{j})!}{\mathbf{m}!\mathbf{j}!}e^{2|\mathbf{m}|(r-s)}e^{-2|\mathbf{j}|s}$$
$$= \|F\|^2_{\mathcal{H},\beta,r}\sum_{\mathbf{j}}\sum_{\mathbf{n}}\frac{\mathbf{n}!}{(\mathbf{n}-\mathbf{j})!\mathbf{j}!}e^{2|\mathbf{n}-\mathbf{j}|(r-s)}e^{-2|\mathbf{j}|s}$$
$$= \|F\|^2_{\mathcal{H},\beta,r}\sum_{\mathbf{n}}\sum_{\mathbf{j}}\frac{\mathbf{n}!}{(\mathbf{n}-\mathbf{j})!\mathbf{j}!}e^{2|\mathbf{n}-\mathbf{j}|(r-s)}e^{-2|\mathbf{j}|s}$$
$$= \|F\|^2_{\mathcal{H},\beta,r}\Big(\sum_{n=0}^{\infty}(e^{-2s}+e^{2(r-s)})^n\Big)^k$$
$$= \|F\|^2_{\mathcal{H}_{\beta,r}}(1-e^{2(r-s)}-e^{-2s})^{-k}$$
$$< \infty.$$

This implies that $\mathcal{H}_\beta(\mathbb{R}^k) \subset \mathcal{F}_\beta(\mathbb{R}^k)$.

Next, we show that $\mathcal{F}_\beta(\mathbb{R}^k) \subset \mathcal{H}_\beta(\mathbb{R}^k)$. Suppose that $F \in \mathcal{F}_\beta(\mathbb{R}^k)$ and let $F(\mathbf{u}) = \sum_{\mathbf{n}} a_{\mathbf{n}}\mathbf{u}^{\mathbf{n}}$. Since

$$\mathbf{u}^{\mathbf{n}} = \sum_{2\mathbf{j}\leq\mathbf{n}}\frac{\mathbf{n}!2^{-\mathbf{j}}}{\mathbf{j}!(\mathbf{n}-2\mathbf{j})!}H_{\mathbf{n}-2\mathbf{j}}(\mathbf{u}),$$

we can express $F(x)$ as follows:

$$F(x) = \sum_{\mathbf{n}} a_{\mathbf{n}} \sum_{2\mathbf{j} \leq \mathbf{n}} \frac{\mathbf{n}!2^{-\mathbf{j}}}{\mathbf{j}!(\mathbf{n}-2\mathbf{j})!} H_{\mathbf{n}-2\mathbf{j}}(\mathbf{u})$$

$$= \sum_{\mathbf{m}} \sum_{\mathbf{j}} a_{\mathbf{m}+2\mathbf{j}} \frac{(\mathbf{m}+2\mathbf{j})!2^{-\mathbf{j}}}{\mathbf{j}!\mathbf{m}!} H_{\mathbf{m}}(\mathbf{u}).$$

Similar calculation yields that for any $r \geq 0$ there exists $s > 0$ with $e^{-2r}+e^{2(s-r)} < 1$ such that

$$\|F\|^2_{\mathcal{H},\beta,r} \leq \|F\|^2_{\mathcal{F},\beta,s}(1 - e^{-2r} - e^{2(s-r)})^{-k} < \infty.$$

This implies that $\mathcal{F}_\beta(\mathbb{R}^k) \subset \mathcal{H}_\beta(\mathbb{R}^k)$. Therefore we show $\mathcal{H}_\beta(\mathbb{R}^k) = \mathcal{F}_\beta(\mathbb{R}^k)$ as sets.

§3.2. Finite Dimensional Hida Distributions of Order β

In this section, the results in §2.2 will be generalized to the spaces $(\mathcal{E}^\beta)^*$, $(\mathcal{E}^\beta)$. Again, the idea is based on [KK 92] and the argument and calculation from [KK 92] are modified to obtain the corresponding results. Recall that from §2.2 that $\langle \cdot, f \rangle$, $f \in E$ is a random variable which has normal distribution with mean zero and variance $|f|_0^2$.

Definition 3.2.1

Let $\{e_1, \cdots, e_k\}$ be linearly independent in E and let V be the linear space spanned by $e_1, \cdots, e_k$. Let $(\mathcal{E}^\beta)^*_V \equiv (\mathcal{E}^\beta)^*$-closure of all polynomials in $\langle \cdot, \vec{e} \rangle$. A elememt Φ in $(\mathcal{E}^\beta)^*$ is called a finite dimensional Hida distribution of order β if $\Phi \in (\mathcal{E}^\beta)^*_V$. In this case, we say that Φ is based on V.

In the following Theorem 3.2.2 and Theorem 3.2.3, the finite dimensional Hida distributions of order β are characterized. These two theorems also characterize such fuctions F that $F \circ \langle \cdot, \xi \rangle \in (\mathcal{E}^\beta)^*$, $\xi \in E$. In other words, the set of the finite dimensional Hida distributions of order β coincides with the set of all compositions of generalized functions in $(\mathcal{H}^\beta)^*(\mathbb{R}^k)$ with Gaussian random variables. We recall the S-transform from §1.2. Let $\Psi \in (\mathcal{E}^\beta)^*$. the S-transform of Psi is given by

$$(S\Psi)(\xi) = \langle\langle \Psi, \exp(\langle \cdot, \xi \rangle - 2^{-1}|\xi|_0^2) \rangle\rangle, \qquad \xi \in \mathcal{E},$$

where we use $\langle\langle \cdot, \cdot \rangle\rangle$ to denote the bilinear pairing between $(\mathcal{E}^\beta)^*$ and$(\mathcal{E}^\beta)$.

Theorem 3.2.2

Let $\{e_1, \cdots, e_k\}$ be orthonormal in E and $V \equiv \text{span}\{e_1, \cdots, e_k\}$.

(1) If $F = \sum_{\mathbf{n}} a_{\mathbf{n}} H_{\mathbf{n}} \in (\mathcal{H}^\beta)^*(\mathbb{R}^k)$, then $\sum_{|\mathbf{n}|=0}^N a_{\mathbf{n}} H_{\mathbf{n}}(\langle \cdot, \vec{e} \rangle)$ converges to an element in $(\mathcal{E}^\beta)^*$ as N goes to infinity;

(2) Let

$$F(\langle \cdot, \vec{e} \rangle) \equiv \lim_{N \to \infty} \sum_{|\mathbf{n}|=0}^N a_{\mathbf{n}} H_{\mathbf{n}}(\langle \cdot, \vec{e} \rangle).$$

Then $F(\langle \cdot, \vec{e} \rangle) \in (\mathcal{E}^\beta)_V^*$ with S-transform given by

$$SF(\langle \cdot, \vec{e} \rangle)(\xi) = (\sigma F)(\langle \xi, \vec{e} \rangle), \qquad \xi \in \mathcal{S}(\mathbb{R}^k),$$

where σF is the σ-transform of F;

(3) The mapping $F \mapsto F(\langle \cdot, \vec{e} \rangle)$ from $(\mathcal{H}^\beta)^*(\mathbb{R}^k)$ into $(\mathcal{E}^\beta)^*$ is continuous.

Proof.

We modify the proof of Theorem 2.2.2 from the paper [KK 92]. Suppose that $F = \sum_{\mathbf{n}} a_{\mathbf{n}} H_{\mathbf{n}} \in (\mathcal{H}^\beta)^*(\mathbb{R}^k)$. Note that the multiple Wiener integral I_n can be regarded as the linear mapping on $E_{\mathbb{C}}^{\widehat{\otimes} n}$ such that

$$I_n(\zeta_1^{\widehat{\otimes} n_1}\widehat{\otimes}\cdots\widehat{\otimes}\zeta_k^{\widehat{\otimes} n_k}) = H_{n_1}(\langle\cdot,\zeta_1\rangle)\cdots H_{n_k}(\langle\cdot,\zeta_k\rangle),$$

where $n_1+\cdots+n_k = n$ and $\zeta_1,\ldots\zeta_k$ are orthonormal. Therefore,

$$\left\|\sum_{\mathbf{n}} a_{\mathbf{n}} H_{\mathbf{n}}(\langle\cdot,\vec{e}\rangle)\right\|_{-q,-\beta}^2 = \sum_{n=0}^{\infty}(n!)^{1-\beta}\left|\sum_{|\mathbf{n}|=n} a_{\mathbf{n}} e_1^{\widehat{\otimes} n_1}\widehat{\otimes}\cdots\widehat{\otimes} e_k^{\widehat{\otimes} n_k}\right|_{-q}^2.$$

By using the following inequality: for any $p\in\mathbb{R}, r\geq 0$,

$$|f|_p \leq \rho^{nr}|f|_{p+r}, \qquad f\in E_{\mathbb{C}}^{\widehat{\otimes} n}$$

where $\rho\equiv\lambda_1^{-1}$ (see §1.1), we obtain

$$\begin{aligned}
\left\|\sum_{\mathbf{n}} a_{\mathbf{n}} H_{\mathbf{n}}(\langle\cdot,\vec{e}\rangle)\right\|_{-q,-\beta}^2 &\leq \sum_{n=0}^{\infty}(n!)^{1-\beta}\rho^{2qn}\left|\sum_{|\mathbf{n}|=n} a_{\mathbf{n}} e_1^{\widehat{\otimes} n_1}\widehat{\otimes}\cdots\widehat{\otimes} e_k^{\widehat{\otimes} n_k}\right|_0^2 \\
&= \sum_{\mathbf{n}}(\mathbf{n}!)^{1-\beta}\rho^{2q|\mathbf{n}|}|a_{\mathbf{n}}|^2 \\
&= \sum_{\mathbf{n}}(\mathbf{n}!)^{1-\beta}e^{-2|\mathbf{n}|q\log\frac{1}{\rho}}|a_{\mathbf{n}}|^2 \\
&= \|F\|^2_{\mathcal{H},-\beta,-q\log\frac{1}{\rho}}.
\end{aligned}$$

The assumption $F\in(\mathcal{H}^\beta)^*(\mathbb{R}^k)$ implies that $\|F\|^2_{\mathcal{H},-\beta,-q\log\frac{1}{\rho}}<\infty$ for some $q>0$. Therefore, the claim (1) holds. And the claim (3) also follows immediately from the above estimation.

Now, it is obvious that $F(\langle\cdot,\vec{e}\rangle)\in(\mathcal{E}^\beta)_V^*$ by the definition of finite dimensional Hida distributions of order β. In order to find its S-transform we use the facts that for $\zeta_1,\ldots,\zeta_k$ orthonormal in $E_{\mathbb{C}}$,

$$(SH_{n_1}(\langle\cdot,\zeta_1\rangle)\cdots H_{n_k}(\langle\cdot,\zeta_k\rangle))(\xi) = \prod_{j=1}^{k}\langle\zeta_j,\xi\rangle^{n_j}, \qquad \xi\in\mathcal{E}.$$

Then we have

$$(SH_{\mathbf{n}}(\langle\cdot,\vec{e}\rangle))(\xi) = \langle\xi,\vec{e}\rangle^{\mathbf{n}}.$$

On the other hand, we recall from §2.1 that

$$(\sigma H_{\mathbf{n}})(\mathbf{u}) = \mathbf{u}^{\mathbf{n}}.$$

Therefore we derive the following

$$(SH_{\mathbf{n}}(\langle\cdot,\vec{e}\rangle))(\xi) = (\sigma H_{\mathbf{n}})(\langle\xi,\vec{e}\rangle)$$

for all $\mathbf{n}$ and $\xi \in \mathcal{E}$. This implies that

$$SF(\langle\cdot,\vec{e}\rangle))(\xi) = (\sigma F)(\langle\xi,\vec{e}\rangle)$$

for all $\xi \in \mathcal{E}$. Thus we get the claim (2). Q.E.D.

Theorem 3.2.3

Let $\{e_1,\cdots,e_k\}$ be orthonormal in E and $V \equiv \text{span}\{e_1,\cdots,e_k\}$. Suppose that $\Phi \in (\mathcal{E}^\beta)^*_V$. Then there exists a function $F \in (\mathcal{H}^\beta)^*(\mathbb{R}^k)$ such that

$$\Phi = F(\langle\cdot,e_1\rangle,\cdots,\langle\cdot,e_k\rangle).$$

Proof.

We modify the proof of Theorem 2.2.3 from the paper [KK 92]. Suppose that $\Phi \in (\mathcal{E}^\beta)^*_V$. Then by the definition of the space $(\mathcal{E}^\beta)^*_V$ we can choose a sequence$\{\Phi_i\}$ of polynomials in $\{\langle\cdot,e_1\rangle,\cdots,\langle\cdot,e_k\rangle\}$ such that Φ_i converges to Φ in $(\mathcal{E}^\beta)^*$ as i goes to infinity. Then, there exists some $p \geq 0$ such that $\|\Phi_i - \Phi\|_{-p,-\beta}$ goes to zero as i goes to infinity. We recall that each Φ_i can be expressed as

$$\Phi_i = \sum I_n(f_n^{(i)}), \qquad f_n^{(i)} \in E_{\mathbb{C}}^{\widehat{\otimes} n}.$$

We want to represent each $f_n^{(i)}$, for fixed i, by means of elements in V. So we choose a collection $\{\xi_1, \cdots, \xi_k\} \subset V$ which are orthonormal with respect to the norm $|\cdot|_{-p}$, i.e., we choose an orthonormal basis for V in E_{-p}. Then we can express each Φ_i as

$$\Phi_i = \sum_n I_n(f_n^{(i)}), \qquad f_n^{(i)} = \sum_{|\mathbf{n}|=n} b_{\mathbf{n}}^{(i)} \xi_1^{\widehat{\otimes} n_1} \widehat{\otimes} \cdots \widehat{\otimes} \xi_k^{\widehat{\otimes} n_k}.$$

Here the summation for Φ_i has only finitely many nonzero terms. So, we have

$$\Phi_i - \Phi_j = \sum_{n=0}^{\infty} I_n \left(\sum_{|\mathbf{n}|=n} (b_{\mathbf{n}}^{(i)} - b_{\mathbf{n}}^{(j)}) \xi_1^{\widehat{\otimes} n_1} \widehat{\otimes} \cdots \widehat{\otimes} \xi_k^{\widehat{\otimes} n_k} \right).$$

and thus we get

$$\|\Phi_i - \Phi_j\|_{-p,-\beta}^2 = \sum_{n=0}^{\infty} (n!)^{1-\beta} \left| \sum_{|\mathbf{n}|=n} (b_{\mathbf{n}}^{(i)} - b_{\mathbf{n}}^{(j)}) \xi_1^{\widehat{\otimes} n_1} \widehat{\otimes} \cdots \widehat{\otimes} \xi_k^{\widehat{\otimes} n_k} \right|_{-p}^2$$

By using the fact that $\{\xi_1, \cdots, \xi_k\}$ is a orthonormal basis for E_{-p} and the definition of the tensor product, we can check the following equation

$$\left| \xi_1^{\widehat{\otimes} n_1} \widehat{\otimes} \cdots \widehat{\otimes} \xi_k^{\widehat{\otimes} n_k} \right|_{-p}^2 = \frac{n_1! \cdots n_k!}{(n_1 + \cdots + n_k)!} = \frac{\mathbf{n}!}{|\mathbf{n}|!}.$$

Then we have

$$\begin{aligned} \|\Phi_i - \Phi_j\|_{-p,-\beta}^2 &= \sum_{n=0}^{\infty} (n!)^{1-\beta} \sum_{|\mathbf{n}|=n} \frac{\mathbf{n}!}{|\mathbf{n}|!} \left| b_{\mathbf{n}}^{(i)} - b_{\mathbf{n}}^{(j)} \right|^2 \\ &= \sum_{\mathbf{n}} \frac{\mathbf{n}!}{(|\mathbf{n}|!)^{\beta}} \left| b_{\mathbf{n}}^{(i)} - b_{\mathbf{n}}^{(j)} \right|^2. \end{aligned}$$

The fact that $\|\Phi_i - \Phi_j\|_{-p,-\beta}^2$ goes to zero implies that $\sum_{\mathbf{n}} \frac{\mathbf{n}!}{(|\mathbf{n}|!)^{\beta}} \left| b_{\mathbf{n}}^{(i)} - b_{\mathbf{n}}^{(j)} \right|^2$ goes to zero as i, j goes to infinity. Thus, there exists the limit, denoted by $b_{\mathbf{n}}$, of $\{b_{\mathbf{n}}^{(i)}\}$ for each $\mathbf{n}$ as i goes to infinity. Thus we can express Φ in terms of elements in V as follows:

$$\Phi = \sum_{n=0}^{\infty} I_n(f_n), \qquad f_n = \sum_{|\mathbf{n}|=n} b_{\mathbf{n}} \xi_1^{\widehat{\otimes} n_1} \widehat{\otimes} \cdots \widehat{\otimes} \xi_k^{\widehat{\otimes} n_k}.$$

And we obtain

$$\|\Phi\|^2_{-p,-\beta} = \sum_{\mathbf{n}} \frac{\mathbf{n}!}{(|\mathbf{n}|!)^\beta} |b_{\mathbf{n}}|^2.$$

In the following, we will express each ξ_i by means of $\{e_1, \cdots, e_k\}$ in order to obtain the coffecients $a_{\mathbf{n}}$ for Hermite polynomials for the expression $F = \sum a_{\mathbf{n}} H_{\mathbf{n}}$ in $(\mathcal{H}^\beta)^*(\mathbb{R}^k)$. Let

$$\xi_i = \sum_{j=1}^{k} \alpha_{i,j} e_j, \qquad i = 1, \cdots, k.$$

Then we have

$$\xi_1^{\widehat{\otimes} n_1} \widehat{\otimes} \cdots \widehat{\otimes} \xi_k^{\widehat{\otimes} n_k} = \sum_{|\mathbf{m}|=|\mathbf{n}|} c_{\mathbf{n},\mathbf{m}} e_1^{\widehat{\otimes} m_1} \widehat{\otimes} \cdots \widehat{\otimes} e_k^{\widehat{\otimes} m_k}.$$

Let $\alpha = \max_{i,j} |\alpha_{i,j}|$. Then we can easily check the following inequality:

$$|c_{\mathbf{n},\mathbf{m}}| \leq \frac{|\mathbf{m}|!}{\mathbf{m}!} \alpha^{|\mathbf{m}|}.$$

So we can represent f_n in terms of $\{e_1, \cdots, e_k\}$ as

$$f_n = \sum_{|\mathbf{m}|=n} a_{\mathbf{m}}\, e_1^{\widehat{\otimes} m_1} \widehat{\otimes} \cdots \widehat{\otimes} e_k^{\widehat{\otimes} m_k}, \qquad a_{\mathbf{m}} = \sum_{|\mathbf{n}|=|\mathbf{m}|} b_{\mathbf{n}} c_{\mathbf{n},\mathbf{m}}.$$

Define $F \equiv \sum_{\mathbf{n}} a_{\mathbf{n}} H_{\mathbf{n}}$. In order to show this F is in $(\mathcal{H}^\beta)^*(\mathbb{R}^k)$, we consider

$$\|F\|^2_{\mathcal{H},-\beta,-t} = \sum_{n=0}^{\infty} e^{-2tn} \sum_{|\mathbf{m}|=n} (\mathbf{m}!)^{1-\beta} |a_{\mathbf{m}}|^2.$$

By using the above equality for $|a_{\mathbf{m}}|$ and inequality for $|c_{\mathbf{n},\mathbf{m}}|$ we have

$$\begin{aligned}
|a_{\mathbf{m}}| &\leq \sum_{|\mathbf{n}|=|\mathbf{m}|} |b_{\mathbf{n}}||c_{\mathbf{n},\mathbf{m}}| \\
&\leq \sum_{|\mathbf{n}|=|\mathbf{m}|} \frac{|\mathbf{m}|!}{\mathbf{m}!} \alpha^{|\mathbf{m}|} |b_{\mathbf{n}}| \\
&= \frac{|\mathbf{m}|!}{\mathbf{m}!} \alpha^{|\mathbf{m}|} \sum_{|\mathbf{n}|=|\mathbf{m}|} |b_{\mathbf{n}}| \frac{(\mathbf{n}!)^{1/2}}{(|\mathbf{n}|!)^{\beta/2}} \frac{(|\mathbf{n}|!)^{\beta/2}}{(\mathbf{n}!)^{1/2}}.
\end{aligned}$$

By the Schwarz inequality, we get

$$|a_{\mathbf{m}}| \leq \frac{|\mathbf{m}|!}{\mathbf{m}!}\alpha^{|\mathbf{m}|}\left(\sum_{|\mathbf{n}|=|\mathbf{m}|}\frac{(|\mathbf{n}|!)^{\beta}}{\mathbf{n}!}\right)^{1/2}\left(\sum_{|\mathbf{n}|=|\mathbf{m}|}|b_{\mathbf{n}}|^2\frac{\mathbf{n}!}{(|\mathbf{n}|!)^{\beta}}\right)^{1/2}.$$

By using the multinomial expansion we get the equality,

$$\sum_{|\mathbf{n}|=n}\frac{(n!)^{\beta}}{\mathbf{n}!} = \frac{k^n}{(n!)^{(1-\beta)}}.$$

From this equality we have

$$|a_{\mathbf{m}}| \leq \frac{|\mathbf{m}|!}{\mathbf{m}!}\alpha^{|\mathbf{m}|}\left(\frac{k^{|\mathbf{m}|}}{(|\mathbf{m}|!)^{1-\beta}}\right)^{1/2}\left(\sum_{|\mathbf{n}|=|\mathbf{m}|}\frac{\mathbf{n}!}{(|\mathbf{n}|!)^{\beta}}|b_{\mathbf{n}}|^2\right)^{1/2}.$$

Thus

$$|a_{\mathbf{m}}|^2 \leq \frac{(|\mathbf{m}|!)^2}{(\mathbf{m}!)^2}\alpha^{2|\mathbf{m}|}\frac{k^{|\mathbf{m}|}}{(|\mathbf{m}|!)^{1-\beta}}\sum_{|\mathbf{n}|=|\mathbf{m}|}\frac{\mathbf{n}!}{(|\mathbf{n}|!)^{\beta}}|b_{\mathbf{n}}|^2.$$

Therefore, by using the above estimation for $a_{\mathbf{m}}$ and noting that $\frac{|\mathbf{m}|!}{\mathbf{m}!} \geq 1$ we obtain

$$\begin{aligned}
\|F\|^2_{\mathcal{H},-\beta,-t} &\leq \sum_{n=0}^{\infty} e^{-2tn}\alpha^{2n}k^n \sum_{|\mathbf{m}|=n}\frac{(\mathbf{m}!)^{1-\beta}}{(|\mathbf{m}|!)^{1-\beta}}\frac{(|\mathbf{m}|!)^2}{(\mathbf{m}!)^2}\sum_{|\mathbf{n}|=|\mathbf{m}|}\frac{\mathbf{n}!}{(|\mathbf{n}|!)^{\beta}}|b_{\mathbf{n}}|^2 \\
&= \sum_{n=0}^{\infty} e^{-2tn}\alpha^{2n}k^n \sum_{|\mathbf{m}|=n}\frac{(|\mathbf{m}|!)^{1+\beta}}{(\mathbf{m}!)^{1+\beta}}\sum_{|\mathbf{n}|=|\mathbf{m}|}\frac{\mathbf{n}!}{(|\mathbf{n}|!)^{\beta}}|b_{\mathbf{n}}|^2 \\
&\leq \sum_{n=0}^{\infty} e^{-2tn}\alpha^{2n}k^n \sum_{|\mathbf{m}|=n}\left(\frac{|\mathbf{m}|!}{\mathbf{m}!}\right)^2\sum_{|\mathbf{n}|=|\mathbf{m}|}\frac{\mathbf{n}!}{(|\mathbf{n}|!)^{\beta}}|b_{\mathbf{n}}|^2 \\
&\leq \sum_{n=0}^{\infty} e^{-2tn}\alpha^{2n}k^n \sum_{|\mathbf{m}|=n}k^n\frac{|\mathbf{m}|!}{\mathbf{m}!}\sum_{|\mathbf{n}|=|\mathbf{m}|}\frac{\mathbf{n}!}{(|\mathbf{n}|!)^{\beta}}|b_{\mathbf{n}}|^2 \\
&= \sum_{n=0}^{\infty}(e^{-2t}\alpha^2k^3)^n\sum_{|\mathbf{n}|=n}\frac{\mathbf{n}!}{(|\mathbf{n}|!)^{\beta}}|b_{\mathbf{n}}|^2.
\end{aligned}$$

Choose $t > 0$ sufficiently large so that $t > \log \alpha k^{3/2}$. Then

$$\begin{aligned}
\|F\|^2_{\mathcal{H},-\beta,-t} &\leq \sum_{n=0}^{\infty} \sum_{|\mathbf{n}|=n} \frac{(\mathbf{n}!)}{(|\mathbf{n}|!)^{\beta}} |b_{\mathbf{n}}|^2 \\
&= \sum_{\mathbf{n}} \frac{(\mathbf{n}!)}{(|\mathbf{n}|!)^{\beta}} |b_{\mathbf{n}}|^2 \\
&= \|\Phi\|^2_{-p,-\beta}.
\end{aligned}$$

Therfore $F \in (\mathcal{H}_t^{\beta})^*(\mathbb{R}^k)$ for the some t. This implies that $F \in (\mathcal{H}^{\beta})^*(\mathbb{R}^k)$. Now, we show the last part of the theorem.

$$\begin{aligned}
F(\langle\cdot,\eta_1\rangle,\cdots,\langle\cdot,\eta_k\rangle) &= \sum_{\mathbf{n}} a_{\mathbf{n}} H_{n_1}(\langle\cdot,\eta_1\rangle)\cdots H_{n_k}(\langle\cdot,\eta_k\rangle) \\
&= \sum_{n=0}^{\infty} \sum_{|\mathbf{m}|=n} a_{\mathbf{n}} I_n(e_1^{\widehat{\otimes} m_1} \widehat{\otimes} \cdots \widehat{\otimes} e_k^{\widehat{\otimes} m_k}) \\
&= \sum_{n=0}^{\infty} I_n(f_n) \\
&= \Phi. \qquad Q.E.D.
\end{aligned}$$

The following two Theorem 3.2.4 and Theorem 3.2.5 we characterize the space $(\mathcal{E}^{\beta})_V \equiv (\mathcal{E}^{\beta})_V^* \cap (\mathcal{E}^{\beta})$ of the finite dimensional Hida distributions of order β in $(\mathcal{E}^{\beta})$. These two theorems characterize such fuctions F that $F \circ \langle\cdot,\xi\rangle \in (\mathcal{E}^{\beta})$, $\xi \in \mathcal{E}$.

Theorem 3.2.4.

Let $\{\eta_1,\cdots,\eta_k\} \subset \mathcal{E}$ be orthonormal in E and $V \equiv \text{span}\{\eta_1,\cdots,\eta_k\}$. If $F \in \mathcal{H}(\mathbb{R}^k)$, then $F(\langle\cdot,\vec{\eta}\rangle) \in (\mathcal{E}^{\beta})_V$. Moreover, the mapping $F \mapsto F(\langle\cdot,\vec{\eta}\rangle)$ from $\mathcal{H}^{\beta}(\mathbb{R}^k)$ into $(\mathcal{E}^{\beta})$ is continuous.

Proof.

We modify the proof of Theorem 2.2.4 from the paper [KK 92]. Suppose that $F = \sum_{\mathbf{n}} a_{\mathbf{n}} H_{\mathbf{n}} \in \mathcal{H}^{\beta}(\mathbb{R}^k)$. We need to consider for any $p \geq 0$

$$\|F(\langle\cdot,\vec{\eta}\rangle)\|_{p,\beta} = \|\sum_{\mathbf{n}} a_{\mathbf{n}} H_{\mathbf{n}}(\langle\cdot,\vec{\eta}\rangle)\|_{p,\beta}.$$

We recall that

$$I_n(\zeta_1^{\widehat{\otimes} n_1}\widehat{\otimes}\cdots\widehat{\otimes}\zeta_k^{\widehat{\otimes} n_k}) = H_{n_1}(\langle\cdot,\zeta_1\rangle)\cdots H_{n_k}(\langle\cdot,\zeta_k\rangle),$$

where $n_1+\cdots+n_k=n$. Thus we have

$$\begin{aligned}
\|F(\langle\cdot,\vec{\eta}\rangle)\|_{p,\beta} &\le \sum_{\mathbf{n}} |a_{\mathbf{n}}|\|H_{\mathbf{n}}(\langle\cdot,\vec{\eta}\rangle)\|_{p,\beta}\\
&= \sum_{\mathbf{n}} |a_{\mathbf{n}}|(|\mathbf{n}|!)^{\frac{1+\beta}{2}}|\eta_1^{\widehat{\otimes} n_1}\widehat{\otimes}\cdots\widehat{\otimes}\eta_k^{\widehat{\otimes} n_k}|_p\\
&= \sum_{\mathbf{n}} |a_{\mathbf{n}}|(|\mathbf{n}|!)^{\frac{1+\beta}{2}}|\eta_1|_p^{n_1}\cdots|\eta_k|_p^{n_k}.
\end{aligned}$$

Then by using the Schwarz inquality we have

$$\begin{aligned}
&\|F(\langle\cdot,\vec{\eta}\rangle)\|_{p,\beta}\\
&= \sum_{\mathbf{n}}\left[\left(\frac{|\mathbf{n}|!}{\mathbf{n}!}\right)^{\frac{1+\beta}{2}}\rho^{q|\mathbf{n}|}|\eta_1|_p^{n_1}\cdots|\eta_k|_p^{n_k}\right]\left[(\mathbf{n}!)^{\frac{1+\beta}{2}}\rho^{-q|\mathbf{n}|}|a_{\mathbf{n}}|\right]\\
&\le \left(\sum_{\mathbf{n}}\left(\frac{|\mathbf{n}|!}{\mathbf{n}!}\right)^{1+\beta}\rho^{2q|\mathbf{n}|}|\eta_1|_p^{2n_1}\cdots|\eta_k|_p^{2n_k}\right)^{1/2}\left(\sum_{\mathbf{n}}(\mathbf{n}!)^{1+\beta}\rho^{-2q|\mathbf{n}|}|a_{\mathbf{n}}|^2\right)^{1/2}\\
&\le \left(\sum_{n=0}^{\infty}\rho^{2qn}(|\eta_1|_p^2+\cdots+|\eta_k|_p^2)^n\right)^{1/2}\|F\|_{\mathcal{H},\beta,q\log\frac{1}{\rho}}.
\end{aligned}$$

Choose $q>0$ such that $\rho^{-2q} > (|\eta_1|_p^2+\cdots+|\eta_k|_p^2)$. Then we obtain

$$\|F(\langle\cdot,\vec{\eta}\rangle)\|_{p,\beta} = (1-\rho^{2q}(|\eta_1|_p^2+\cdots+|\eta_k|_p^2))^{-1/2}\|F\|_{\mathcal{H},\beta,q\log\frac{1}{\rho}}.$$

Therefore we have for all $p\ge 0$, $F(\langle\cdot,\vec{\eta}\rangle)\in(E_{p,\beta})$. Thus $F(\langle\cdot,\vec{\eta}\rangle)\in(\mathcal{E}^\beta)$. On the other hand, from Theorem 3.2.2 we note that $F(\langle\cdot,\vec{\eta}\rangle)\in(\mathcal{E}^\beta)_V^*$. Therefore $F(\langle\cdot,\vec{\eta}\rangle)\in(\mathcal{E}^\beta)_V$. Moreover the above estimation yields the continuity of mapping $F\mapsto F(\langle\cdot,\vec{\eta}\rangle)$ from $\mathcal{H}^\beta(\mathbb{R}^k)$ into $(\mathcal{E}^\beta)$. Q.E.D.

Theorem 3.2.5.

Let $\{\eta_1,\cdots,\eta_k\}\subset\mathcal{E}$ be orthonormal in E and $V\equiv\operatorname{span}\{\eta_1,\cdots,\eta_k\}$. If $\phi\in(\mathcal{E}^\beta)_V$, then there exists a function $F\in\mathcal{H}(\mathbb{R}^k)$ such that

$$\phi = F(\langle\cdot,\eta_1\rangle,\cdots,\langle\cdot,\eta_k\rangle).$$

Proof.

We modify the proof of Theorem 2.2.5 from the paper [KK 92]. Suppose that $\phi \in (\mathcal{E}^\beta)_V$. By Theorem 3.2.3 there exists a function $F \in (\mathcal{H}^\beta)^*(\mathbb{R}^k)$ such that

$$\phi = F(\langle \cdot, \eta_1 \rangle, \cdots, \langle \cdot, \eta_k \rangle).$$

We will see that this F is in $\mathcal{H}^\beta(\mathbb{R}^k)$ by estimating the norm $\|F\|_{\mathcal{H},\beta,t}$ for any $t \geq 0$. Recall that F can be represented by $\sum a_{\mathbf{n}} H_{\mathbf{n}}$. Thus we have

$$\phi = \sum a_{\mathbf{n}} H_{n_1}(\langle \cdot, \eta_1 \rangle), \cdots, H_{n_k}(\langle \cdot, \eta_k \rangle).$$

Then we get

$$\|F\|_{\mathcal{H},\beta,t} = \sum_{n=0}^{\infty} e^{2nt} \sum_{|\mathbf{m}|=n} (\mathbf{m}!)^{1+\beta} |a_{\mathbf{m}}|^2.$$

As in the proof of Theorem 3.2.3 ϕ can be expressed as

$$\phi = \sum_{n=0}^{\infty} I_n(f_n), \qquad f_n = \sum_{|\mathbf{m}|=n} a_{\mathbf{m}} \eta_1^{\widehat{\otimes} m_1} \widehat{\otimes} \cdots \widehat{\otimes} \eta_k^{\widehat{\otimes} m_k}.$$

By using the fact that $\{\eta_1, \ldots \eta_k\}$ are orthonormal in E and the definition of the tensor product, we can check

$$\left| \eta_1^{\widehat{\otimes} m_1} \widehat{\otimes} \cdots \widehat{\otimes} \eta_k^{\widehat{\otimes} m_k} \right|_0^2 = \frac{m_1! \cdots m_k!}{(m_1 + \cdots + m_k)!} = \frac{\mathbf{m}!}{|\mathbf{m}|!}.$$

Thus we have

$$|f_n|_0^2 = \sum_{|\mathbf{m}|=n} |a_{\mathbf{m}}|^2 \frac{\mathbf{m}!}{|\mathbf{m}|!}.$$

Then we get the following estimation by using the fact that $\frac{\mathbf{m}!}{|\mathbf{m}|!} \leq 1$ and the inequality $|f|_p \leq \rho^{nr}|f|_{p+r}, \quad f \in E_{\mathbb{C}}^{\widehat{\otimes} n}$ for any $p \in \mathbb{R}, r \geq 0$:

$$
\begin{aligned}
\|F\|^2_{\mathcal{H},\beta,t} &= \sum_{n=0}^{\infty} e^{2nt} \sum_{|\mathbf{m}|=n} |a_{\mathbf{m}}|^2 (\mathbf{m}!)^{1+\beta} \\
&= \sum_{n=0}^{\infty} (n!)^{1+\beta} e^{2nt} \sum_{|\mathbf{m}|=n} |a_{\mathbf{m}}|^2 \frac{(\mathbf{m}!)^{1+\beta}}{(|\mathbf{m}|!)^{1+\beta}} \\
&= \sum_{n=0}^{\infty} (n!)^{1+\beta} e^{2nt} \sum_{|\mathbf{m}|=n} |a_{\mathbf{m}}|^2 \frac{\mathbf{m}!}{|\mathbf{m}|!} \left(\frac{\mathbf{m}!}{|\mathbf{m}|!}\right)^{\beta} \\
&\leq \sum_{n=0}^{\infty} (n!)^{1+\beta} e^{2nt} \sum_{|\mathbf{m}|=n} |a_{\mathbf{m}}|^2 \frac{\mathbf{m}!}{|\mathbf{m}|!} \\
&= \sum_{n=0}^{\infty} (n!)^{1+\beta} e^{2nt} |f_n|_0^2 \\
&\leq \sum_{n=0}^{\infty} (n!)^{1+\beta} e^{2nt} \rho^{2pn} |f_n|_p^2.
\end{aligned}
$$

Thus for any $t \geq 0$ choose $p > 0$ such that $e^{2t}\rho^{2p} < 1$. Then we obtain

$$
\begin{aligned}
\|F\|^2_{\mathcal{H},\beta,t} &\leq \sum_{n=0}^{\infty} (n!)^{1+\beta} |f_n|_p^2 \\
&= \|\phi\|^2_{p,\beta}.
\end{aligned}
$$

Hence we showed that $F \in \mathcal{H}_t^{\beta}(\mathbb{R}^k)$ for any $t \geq 0$ and thus by the way of the construction of the space $\mathcal{H}^{\beta}(\mathbb{R}^k)$, $F \in \mathcal{H}^{\beta}(\mathbb{R}^k)$. Q.E.D.

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